Pre-lab 4: Complex Impedance and Circuit Phase

Doing reactive circuit math involves complex quantities (to capture magnitude and phase), which is necessarily, uh, *complex*. If you are not very comfortable with complex numbers, don't worry. We will also employ a more visual method in class (the method of "phasor diagrams").

Time Domain vs Frequency Domain

Last time we dealt with capacitors in the *time domain*, for example what could be considered the circuit designer's definition: $I = C\,\mathrm{d}V/\mathrm{d}t$ (a physicist would prefer $Q = CV$). That let us learn about charging and discharging (via a constant current or through a resistor), and about how to make an approximate integrator or differentiator.

Now we are going to take a complementary view, to see how to characterize the capacitor in terms of how it behaves when subjected to a sine wave of some frequency.

Frequency Domain

Circuits with capacitors are more complicated than the resistive circuits we talked about earlier, in that their behavior depends on frequency: A "voltage divider" containing capacitors will have a frequency-dependent division ratio. Circuits containing capacitors "corrupt" input waveforms such as square waves, as we saw in the previous lab.

However, both resistors and capacitors are *linear* devices, meaning that the amplitude of the output waveform, whatever its shape, increases exactly in proportion to the input waveform's amplitude. This linearity has many consequences, the most important of which is probably the following:

> *The output of a linear circuit, driven with a sine wave at some frequency f, is itself a sine wave at the same frequency (with, at most, changed amplitude and phase).*

Because of this remarkable property of circuits containing resistors and capacitors (and inductors and linear amplifiers), it is particularly convenient to analyze any such circuit by asking how the output voltage (amplitude and phase) depends on the input voltage, *for sine-wave input at a single frequency*, even though this may not be the intended use. A graph of the resulting *frequency response*, in which the ratio of output to input is plotted for each sine-wave frequency, is useful for thinking about many kinds of waveforms.

Perhaps the most familiar example of thinking of complicated waveforms as collections of sine waves is music. A song might have three notes playing at once. We could make a graph of sound pressure vs time, and see a complicated wiggling waveform, or we could think in frequency space, and consider the sound to be a combination of three notes (the sum of three sine waves). A linear circuit might affect each of these individual waves differently, as we see below. For example, a "bass boost" would make the lowest frequencies louder, without affecting the highest frequencies, and you would hear a sound with more bass.

Impedance

As we will see, it is possible to generalize Ohm's law, replacing the word "resistance" with "impedance," in order to describe any circuit containing these linear passive devices (resistors, capacitors, and inductors). You could think of the subject of impedance (generalized resistance) as Ohm's law for circuits that include capacitors and inductors.

Impedance (symbol: Z) is the "generalized resistance". If we apply a sine-wave voltage

$$V(t) = V_0 \sin(\omega t) \tag{1}$$

across a resistor R, the current I through it is

$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R} \sin(\omega t). \tag{2}$$

We define the impedance Z_R of this resistor by the voltage/current ratio

$$Z_R = \frac{V(t)}{I(t)} = R, \tag{3}$$

which is simply its resistance. (That was easy.) For a capacitor, we know $I = C\, \mathrm{d}V/\mathrm{d}t$. For the same voltage, it gives

$$I(t) = CV_0 \frac{\mathrm{d}\sin(\omega t)}{\mathrm{d}t} = CV_0\, \omega \cos(\omega t). \tag{4}$$

We now have a little problem: the voltage/current ratio is not a constant:

$$\frac{V(t)}{I(t)} = \frac{V_0 \sin(\omega t)}{CV_0 \cos(\omega t)} = \frac{\tan(\omega t)}{C}. \tag{5}$$

This is where you recall what you may have seen in Physics 15C and replace the sine waves with a complex exponential:[1]

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t). \tag{6}$$

That allows us to write

$$V(t) = V_0\, e^{j\omega t}, \tag{7}$$

$$I(t) = C\frac{\mathrm{d}V(t)}{\mathrm{d}t} = j\omega C V_0\, e^{j\omega t}. \tag{8}$$

We can now write the impedance Z_C of the capacitor C as

$$Z_C = \frac{V(t)}{I(t)} = \frac{1}{j\omega C}. \tag{9}$$

Circuits and Phase

The impedance of a capacitor is imaginary. What does that mean? It means that the voltage and the current are out of phase with each other by 90°. We already know that $V(t) \sim \sin(\omega t)$ and $I(t) \sim \cos(\omega t)$. Above, we wrote this in complex exponential form which I will expand to explicitly express the phase difference ($\pi/2$) between V and I.

$$V(t) = V_0\, e^{j\omega t} \tag{10}$$

$$I(t) = j\omega C V_0\, e^{j\omega t} = e^{j\pi/2} \omega C V_0\, e^{j\omega t} = CV_0\, e^{j(\omega t + \pi/2)} \tag{11}$$

[1] In electronics, we use j, not i, for $\sqrt{-1}$ to avoid confusion with the current. If you haven't seen this before, don't worry. We will build an intuitive understanding of complex representations without doing too much math.

There is a $\pi/2$ phase shift between I and V, meaning that V is leading I by 90°. This is expected for $\sin$ and $\cos$ functions.

More physically, it means that in order to build up voltage on the capacitor, it needs time to charge first: there must be a current **before** there can be a voltage, so the I and V waveforms will not line up with each other.

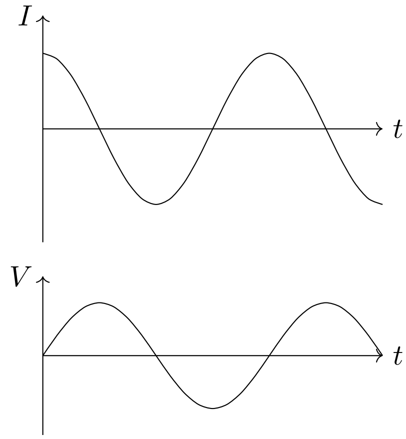


Figure 1: Current I in a capacitor leads the sinusoidal voltage V by 90°.

A bit of terminology: a circuit element for which the voltage and current are 90° out of phase — such as a capacitor (or an inductor) — is called *reactive*; Resistors, with voltage and current always in phase, are *resistive*. In general, in a circuit that combines resistive and reactive components, the voltage and current at some place will have some in-between phase relationship, described by a complex impedance

$$Z = R + jX, \tag{12}$$

where R is the resistance and X is the *reactance*. In a nutshell, the magnitude $|Z|$ of the impedance gives the ratio of amplitudes of voltage to current, and the phase $\arg(Z)$ of the impedance gives the phase angle between current and voltage. However, you'll see statements like "the impedance of the capacitor at this frequency is ..." The reason you don't have to use the word "reactance" in such a case is that impedance covers everything. In fact, you frequently use the word "impedance" even when you know it's a resistance you're talking about; you say "the source impedance" or "the output impedance" when you mean the Thévenin equivalent resistance of some source. The same holds for "input impedance."

Reactance of a Capacitor

The magnitude of the impedance of a capacitor

$$|Z_C| = \frac{1}{\omega C} = \frac{1}{2\pi f C} \tag{13}$$

is inversely proportional to the capacitance and the frequency. (Remember the angular frequency ω is related to the 'ordinary' frequency f by $\omega = 2\pi f$.)

This means that a larger capacitance has a smaller reactance. And this makes sense, because, for example, if you double the value of a capacitor, it takes twice as much current to charge and discharge it through the same voltage swing in the same time (recall $I = C\,\mathrm{d}V/\mathrm{d}t$). For the same reason the reactance decreases as you increase the frequency — doubling the frequency (while holding V constant) doubles the rate of change of voltage and therefore requires that you double the current, thus half the reactance.

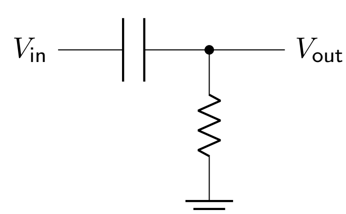
So, roughly speaking, **we can think of a capacitor as a "frequency-dependent resistor."** Sometimes that's good enough, sometimes it isn't. We'll look at a few circuits in which this simplified view gets us reasonably good results, and provides nice intuition. later we'll fix it up, using the correct complex algebra, to get a precise result.

Pre-lab 4 Hand-in

Name: ______________________________

Pre-lab 4.1 Practice drawing a "phasor" diagram of the voltage and current across (a) a resistor and (b) a capacitor. To draw a phasor diagram, make a set of axes with imaginary components in the vertical and real components in the horizontal. The phasor is a vector that expresses the magnitude and phase of your signal. So you will want an arrow that is the length of the amplitude of the voltage signal at an angle given by its phase, and the same for current.

Pre-lab 4.2 Capacitors are reactive and resistors are resistive. But we can get something in between when we consider an RC circuit, or a capacitor loaded by a resistor. Use what you know about voltage dividers to calculate the output voltage of the circuit below. How does the phase of the output voltage compare to the phase of the input voltage? (*Hint: you can represent both signals as phasors in the complex plane.*)



From the last lab:

Pre-lab 4.3 Suppose your physics experiment contains a sensor that produces square voltage pulses at 10 kHz, and has an output impedance R_s of 50 Ω. You want to *differentiate* the signal in order to trigger another part of the experiment on the edges of the pulse. Which of the circuits below would you use to produce a differentiated output? Explain your choice and make a *qualitative* sketch of the expected output.