Pre-lab 3: Capacitors; Integrator and Differentiator

Capacitors — In the Time Domain

Now that we've entered the world of *changing* voltages and currents, or "signals," we encounter two very interesting circuit elements that are useless in purely dc circuits: *capacitors* and *inductors*. As we'll see, these humble devices, combined with resistors, complete the triad of passive linear circuit elements that form the basis of nearly all circuitry. Capacitors, in particular, are essential in nearly every circuit application. They are used for waveform generation, filtering, energy storage, and blocking and bypass applications. They are used in integrators and differentiators. In combination with inductors, they make possible sharp filters for separating desired signals from background.

In today's lab we'll look at capacitors in the *time domain*, and, next time, also in the *frequency domain*; these complementary views are needed to understand *waveform generation* and *filtering*, respectively.

Figure 1: Schematic symbols for a regular (left) capacitor and a polarized (right) capacitor. The curved electrode of the latter indicates the negative terminal.

Capacitance

A capacitor (Fig. 1) is a device that has two wires sticking out of it and has the property

$$Q = CV \tag{1}$$

Its basic form is a pair of closely-spaced metal plates, separated by some insulating material. The unit of the capacitance C is *farad* (symbol "F"), defined by coulomb per volt. A capacitor of C farads with V volts across its terminals has Q coulombs of stored charge on one plate, and $-Q$ on the other.

For the simple parallel-plate capacitor, with separation d and plate area A (and with the spacing d much less than the dimensions of the plates), the capacitance C is given by

$$C = \frac{\kappa\epsilon_0 A}{d}, \tag{2}$$

where κ is the dielectric constant of the insulator and $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$ is the permittivity of free space. It takes a lot of area, and tiny spacing, to make the sort of capacitances commonly used in circuits.[1] For example, a pair of 1 cm square plates separated by 1 mm makes a capacitor of slightly less than 10^{-12} F (a *picofarad*); you'd need 100,000 of them just to create a 0.1 μF capacitor we will routinely use in this course. Ordinarily you don't need to calculate capacitances, because you buy a capacitor as an electronic component.

Voltage vs. Current

Taking the time derivative of the defining equation (1) above, you get

$$I = C\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{3}$$

[1] And it doesn't hurt to have a high dielectric constant as well: air has $\kappa = 1$, but plastic films have $\kappa = 2.1$ (polypropylene) or 3.1 (polyester). Certain ceramics are popular among capacitor-makers: $\kappa = 45$ ("C0G" type) or 3000 ("X7R" type).

So a capacitor is more complicated than a resistor: the current I is not simply proportional to the voltage V, but rather to the rate of change $\mathrm{d}V/\mathrm{d}t$ of voltage. If you change the voltage across a 1 F capacitor by 1 V/s, you are supplying 1 A of current. Conversely, if you supply 1 A of current, its voltage changes by 1 V/s.

A farad is an enormous capacitance, and you usually deal in *microfarads* (μF), *nanofarads* (nF), or *picofarads* (pF). For instance, if you supply a current of 1 mA to 1 μF, the voltage will rise at

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{I}{C} = \frac{1\,\text{mA}}{1\,\mu\text{F}} = 1\,\text{mV/s}.$$

A 10 ms pulse of this current will increase the voltage across the capacitor by 10 volts (Fig. 2).

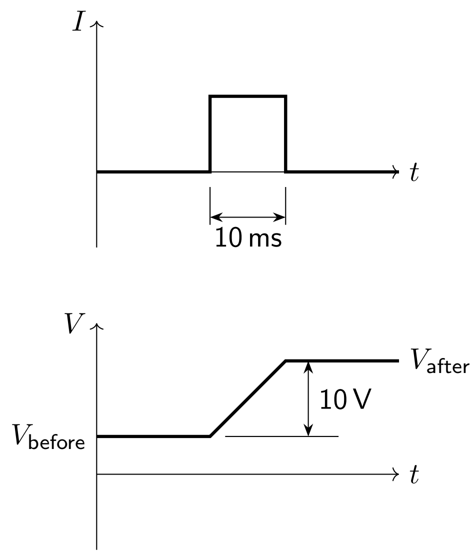


Figure 2: The voltage V across a capacitor changes when a current I flows through it.

The current that flows in a capacitor during charging ($I = C\,\mathrm{d}V/\mathrm{d}t$) has some unusual features. Unlike resistive current, it's not proportional to voltage, but rather to the rate of change (the "time derivative") of voltage. Furthermore, unlike the situation in a resistor, the power (V times I) associated with capacitive current is not turned into heat, but is stored as energy in the capacitor's internal electric field. You get all that energy back when you discharge the capacitor.

Energy

When you charge up a capacitor, you're supplying energy. The capacitor doesn't get hot; instead it stores the energy in its internal electric fields. It's an easy exercise to discover for yourself that the amount of stored energy in a charged capacitor is just

$$U_C = \frac{1}{2}CV^2 \tag{4}$$

where U_C is in joules for C in farads and V in volts. This is an important result, and you will derive it in Pre-lab 3.1.

Capacitors in Parallel and in Series

The capacitance of several capacitors in parallel is the sum of their individual capacitances. This is easy to see: Put voltage V across the parallel combination; then

$$\begin{aligned} C_{\text{total}}V = Q_{\text{total}} &= Q_1 + Q_2 + Q_3 + \cdots \\ &= C_1V + C_2V + C_3V + \cdots \\ &= (C_1 + C_2 + C_3 + \cdots)V \end{aligned}$$

or

$$C_{\text{total}} = C_1 + C_2 + C_3 + \cdots \qquad \text{(parallel)} \tag{5}$$

For capacitors in series, the formula is like that for resistors in parallel:

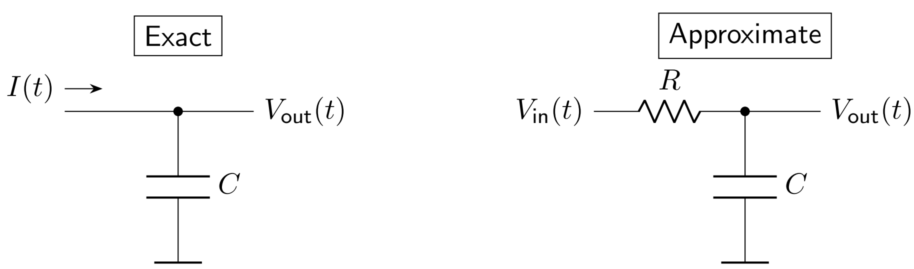
$$C_{\text{total}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots} \qquad \text{(series)} \tag{6}$$

Two capacitors C_1 and C_2 in series gives, for example,

$$C_{\text{total}} = \frac{C_1C_2}{C_1 + C_2} \qquad \text{(two in series)} \tag{7}$$

Integrator and Differentiator

So, capacitors are calculus-friendly ($I = C\ \mathrm{d}V/\mathrm{d}t$), which lets us do cool things like create the derivative or integral of an input waveform. Look at Figure 3, left: since the current through a capacitor is proportional to the time derivative of the voltage across it, it naturally (and effortlessly!) creates a voltage that is the time integral of the current through it. But whatever is a *current* waveform? We'll learn about such a beast, later; but for now we can convert the circuit into an approximate integrator (and you will, in lab), by adding a resistor, as in the right-hand circuit.

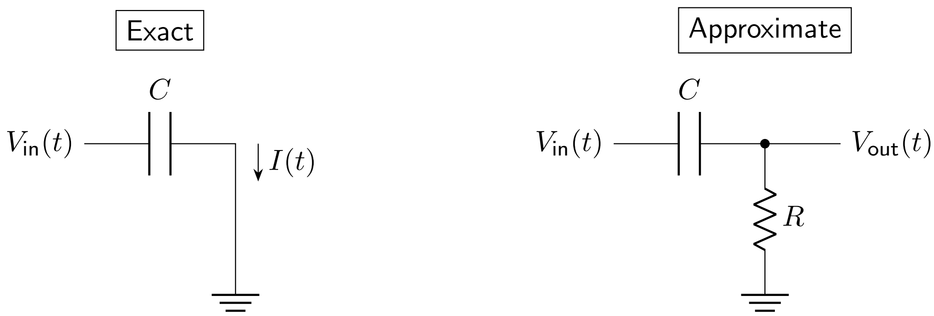


$$I = C\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t},$$
So $V_{\text{out}} = \dfrac{1}{C}\displaystyle\int I(t)\mathrm{d}t.$

$$I = C\frac{\mathrm{d}V_{\text{out}}}{\mathrm{d}t} = \frac{V_{\text{in}} - V_{\text{out}}}{R}.$$
If $|V_{\text{out}}| \ll |V_{\text{in}}|$, then $I \approx \dfrac{V_{\text{in}}}{R}$.
So $V_{\text{out}} \approx \dfrac{1}{RC}\displaystyle\int V_{\text{in}}(t)\mathrm{d}t.$

Figure 3: The voltage across a capacitor is the time integral of the current through it. The circuit on the right is an approximate integrator with a *voltage* input signal.

Analogously, a lone capacitor acts as a differentiator: the current through it is proportional to the time derivative of the voltage across it, as in the left-hand circuit of Figure 4. Problem is, there's no *output* signal! We fix that, approximately, by adding a current-sensing resistor, as in the right-hand circuit.



$$I(t) = C\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t}.$$

$$I(t) = C\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{in}} - V_{\text{out}}).$$

If $\dfrac{\mathrm{d}}{\mathrm{d}t}|V_{\text{out}}| \ll \dfrac{\mathrm{d}}{\mathrm{d}t}|V_{\text{in}}|$,

then $V_{\text{out}}(t) = IR \approx RC\dfrac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t}$..

Figure 4: The current through a capacitor is the derivative of the voltage across it. The circuit on the right is an approximate differentiator with a *voltage* output signal.

Pre-lab 3 Hand-in

Name: ______________________________

Pre-lab 3.1 Derive the energy stored in a capacitor in equation 4. Imagine charging up a capacitor of capacitance C by flowing current I, starting from $V = 0\,\mathrm{V}$ to some final voltage V_f. The result won't depend on whether I is constant or time-varying, but you are welcome to assume constant current for simplicity. At any instant, the rate of flow of energy into the capacitor is $\mathrm{d}U_C/\mathrm{d}t = VI$; so you need to integrate $\mathrm{d}U_C = VI\mathrm{d}t$ from start to finish. Take it from there.

Pre-lab 3.2 Derive the formula for the capacitance of two capacitors in series in equation 7. Hint: Because there is no other connection to the point where the two capacitors are connected together, they must have equal stored charges.

The following problems relate to the last lab. Since this lab was canceled, you are not required to complete these.

Pre-lab 3.3 A function generator with source resistance (i.e., output impedance) $R_{\text{s}} = 50\,\Omega$ drives a load of input resistance (i.e., input impedance) R_{in} with a square wave of $2\,\text{V}$ amplitude peak-to-peak. Suppose you measure the voltage across the load. What is the smallest load resistance R_{in} for which you will measure at least $1.8\,\text{V}$ peak-to-peak?

Pre-lab 3.4 What is the ideal value for the input/output impedance of the following devices? You can make general statements and compare to the impedance of the device in question.

1. Input impedance of a voltmeter.

2. Output impedance of a signal source, like a function generator.

3. Output impedance of a power supply.