Pre-lab 5: Capacitors in the Frequency Domain; Filters

An apology: this pre-lab is long. Doing reactive circuit math involves complex quantities (to capture magnitude and phase), which is necessarily, uh, *complex*. If you are not very comfortable with complex numbers, don't worry. We will also employ a more visual method in class (the method of "phasor diagrams").

Detour: Decibels

Decibels (dB) are logarithmic unit system for describing ratios. A 'decibel' is one tenth of a 'bel', which is the base-10 logarithm of the power ratio. The decibel is defined in terms of power, which is proportional to the square of the voltage amplitude of a signal. That means

$$\text{Ratio in dB} = 10\log_{10}\left(\frac{P}{P_0}\right) = 20\log_{10}\left(\frac{A}{A_0}\right), \tag{1}$$

where P/P_0 and A/A_0 are ratios of power and amplitude, respectively. The table below shows the most important decibel numbers to remember: those that correspond to factors of 2 or 10 in power or amplitude.

	20 dB	6 dB	3 dB	0 dB	-3 dB	-6 dB	-20 dB
Power ratio P/P_0	100	4	2	1	1/2	1/4	1/100
Amplitude ratio A/A_0	10	2	$\sqrt{2}$	1	$1/\sqrt{2}$	1/2	1/10

Table 1: Decibel reference, using the approximation $\log_{10}(2) \approx 0.3$. You will probably end up memorizing most of these values.

Always keep in mind that dB describe *relative* values. When we speak of "attenuation" or "gain" in dB, the ratio in question is $V_{\text{out}}/V_{\text{in}}$ or $P_{\text{out}}/P_{\text{in}}$. So, a device with voltage gain of $G = V_{\text{out}}/V_{\text{in}} = 10$ can also be described as having a gain of 20 dB.

Frequency Domain

Circuits with capacitors are more complicated than the resistive circuits we talked about earlier, in that their behavior depends on frequency: A "voltage divider" containing capacitors will have a frequency-dependent division ratio. Circuits containing capacitors "corrupt" input waveforms such as square waves, as we saw in the previous lab.

However, both resistors and capacitors are *linear* devices, meaning that the amplitude of the output waveform, whatever its shape, increases exactly in proportion to the input waveform's amplitude. This linearity has many consequences, the most important of which is probably the following:

> *The output of a linear circuit, driven with a sine wave at some frequency f, is itself a sine wave at the same frequency (with, at most, changed amplitude and phase).*

Because of this remarkable property of circuits containing resistors and capacitors (and inductors and linear amplifiers), it is particularly convenient to analyze any such circuit by asking how the output voltage (amplitude and phase) depends on the input voltage, *for sine-wave input at a single frequency*, even though this may not be the intended use. A graph of the resulting *frequency response*, in which the ratio of output to input is plotted for each sine-wave frequency, is useful for thinking about many kinds of waveforms.

Perhaps the most familiar example of thinking of complicated waveforms as collections of sine waves is music. A song might have three notes playing at once. We could make a graph of sound pressure vs time, and see a complicated wiggling waveform, or we could think in frequency space, and consider the sound to be a combination of three notes (the sum of three sine waves). A linear circuit might affect each of these individual waves differently, as we see below. For example, a "bass boost" would make the lowest frequencies louder, without affecting the highest frequencies, and you would hear a sound with more bass.

Impedance

As we will see, it is possible to generalize Ohm's law, replacing the word "resistance" with "impedance," in order to describe any circuit containing these linear passive devices (resistors, capacitors, and inductors). You could think of the subject of impedance (generalized resistance) as Ohm's law for circuits that include capacitors and inductors.

Impedance (symbol: Z) is the "generalized resistance". If we apply a sine-wave voltage

$$V(t) = V_0 \sin(\omega t) \tag{2}$$

across a resistor R, the current I through it is

$$I(t) = \frac{V(t)}{R} = \frac{V_0}{R} \sin(\omega t). \tag{3}$$

We define the impedance Z_R of this resistor by the voltage/current ratio

$$Z_R = \frac{V(t)}{I(t)} = R, \tag{4}$$

which is simply its resistance. (That was easy.) For a capacitor, we know $I = C\, \mathrm{d}V/\mathrm{d}t$. For the same voltage, it gives

$$I(t) = CV_0 \frac{\mathrm{d}\sin(\omega t)}{\mathrm{d}t} = CV_0\, \omega \cos(\omega t). \tag{5}$$

We now have a little problem: the voltage/current ratio is not a constant:

$$\frac{V(t)}{I(t)} = \frac{V_0 \sin(\omega t)}{CV_0 \cos(\omega t)} = \frac{\tan(\omega t)}{C}. \tag{6}$$

This is where you recall what you may have seen in Physics 15C and replace the sine waves with a complex exponential:[1]

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t). \tag{7}$$

That allows us to write

$$V(t) = V_0\, e^{j\omega t}, \tag{8}$$

$$I(t) = C \frac{\mathrm{d}V(t)}{\mathrm{d}t} = j\omega C V_0\, e^{j\omega t}. \tag{9}$$

[1] In electronics, we use j, not i, for $\sqrt{-1}$ to avoid confusion with the current. If you haven't seen this before, don't worry. We will build an intuitive understanding of complex representations without doing too much math.

We can now write the impedance Z_C of the capacitor C as

$$Z_C = \frac{V(t)}{I(t)} = \frac{1}{j\omega C}. \tag{10}$$

The impedance of a capacitor is imaginary. What does that mean? It means that the voltage and the current are out of phase with each other by 90°. More physically, it means that in order to build up voltage on the capacitor, it needs time to charge first: there must be a current **before** there can be a voltage, so the I and V waveforms will not line up with each other.

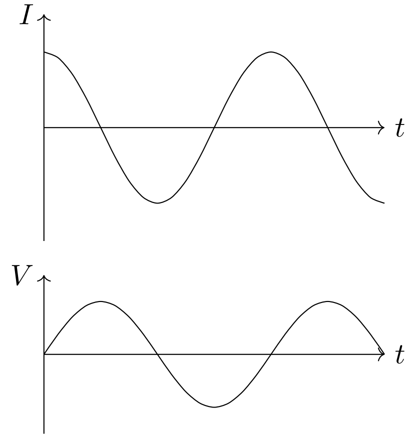


Figure 1: Current I in a capacitor leads the sinusoidal voltage V by 90°.

A bit of terminology: a circuit element for which the voltage and current are 90° out of phase — such as a capacitor (or an inductor) — is called *reactive*; Resistors, with voltage and current always in phase, are *resistive*. In general, in a circuit that combines resistive and reactive components, the voltage and current at some place will have some in-between phase relationship, described by a complex impedance

$$Z = R + jX, \tag{11}$$

where R is the resistance and X is the *reactance*. In a nutshell, the magnitude $|Z|$ of the impedance gives the ratio of amplitudes of voltage to current, and the phase $\arg(Z)$ of the impedance gives the phase angle between current and voltage. However, you'll see statements like "the impedance of the capacitor at this frequency is ..." The reason you don't have to use the word "reactance" in such a case is that impedance covers everything. In fact, you frequently use the word "impedance" even when you know it's a resistance you're talking about; you say "the source impedance" or "the output impedance" when you mean the Thévenin equivalent resistance of some source. The same holds for "input impedance."

Reactance of a Capacitor

The magnitude of the impedance of a capacitor

$$|Z_C| = \frac{1}{\omega C} = \frac{1}{2\pi f C} \tag{12}$$

is inversely proportional to the capacitance and the frequency. (Remember the angular frequency ω is related to the 'ordinary' frequency f by $\omega = 2\pi f$.)

This means that a larger capacitance has a smaller reactance. And this makes sense, because, for example, if you double the value of a capacitor, it takes twice as much current to charge and discharge it through the same voltage swing in the same time (recall $I = C\,\mathrm{d}V/\mathrm{d}t$). For the same reason the reactance decreases as you increase the frequency — doubling the frequency (while holding V constant) doubles the rate of change of voltage and therefore requires that you double the current, thus half the reactance.

So, roughly speaking, **we can think of a capacitor as a "frequency-dependent resistor."** Sometimes that's good enough, sometimes it isn't. We'll look at a few circuits in which this simplified view gets us reasonably good results, and provides nice intuition. later we'll fix it up, using the correct complex algebra, to get a precise result.

RC Low-pass Filter (approximate)

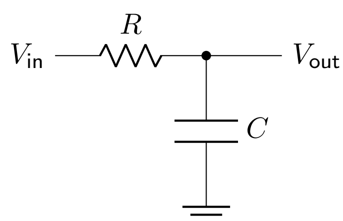


Figure 2: RC low-pass filter.

The circuit in Fig. 2 is called a *low-pass filter*, because it passes low frequencies and blocks high frequencies. If you think of it as a frequency-dependent voltage divider, this makes sense: the lower leg of the divider (the capacitor) has a decreasing reactance with increasing frequency, so the ratio of $V_{\text{out}}/V_{\text{in}}$ decreases accordingly.

To treat this fully, we would use our favorite equation, the voltage divider equation, replacing R with Z. But Z is complex! For now, we will make an approximation, and replace the Rs with the *magnitude* of the Zs of each component:

$$\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{|Z_C|}{|Z_R| + |Z_C|} = \frac{X_C}{R + X_C} = \frac{1/\omega C}{R + 1/\omega C} = \frac{1}{1 + \omega RC} \qquad \text{(approximate!)}$$

We've plotted that ratio in Fig. 3 (and also that of its cousin, the *highpass filter*), along with their exact results. You can see that the circuit passes low frequencies fully (because at low frequencies the capacitor's reactance is very high, so it's like a divider with a smaller resistor atop a larger one), and that it blocks high frequencies. In particular, the crossover from "pass" to "block" occurs at a frequency ω_0 at which the capacitor's reactance ($1/\omega_0 C$) is equal to the resistance R: $\omega_0 = 1/RC$. Expressed in ordinary (not angular) frequency,

$$f_0 = \frac{1}{2\pi RC}.$$

At frequencies well beyond the crossover (where the product $\omega RC \gg 1$), the output decreases inversely proportional to the frequency; that makes sense because the reactance of the capacitor, already much smaller than R, continues dropping as $1/\omega$. It's worth noting that, even with our "ignoration of phase shifts," the equation (and graph) for the ratio of voltages is quite accurate at both low and high frequencies, and is only modestly in error around the crossover frequency, where the correct ratio is $V_{\text{out}}/V_{\text{in}} = 1/\sqrt{2} \approx 0.7$, rather than the 0.5 that we got.

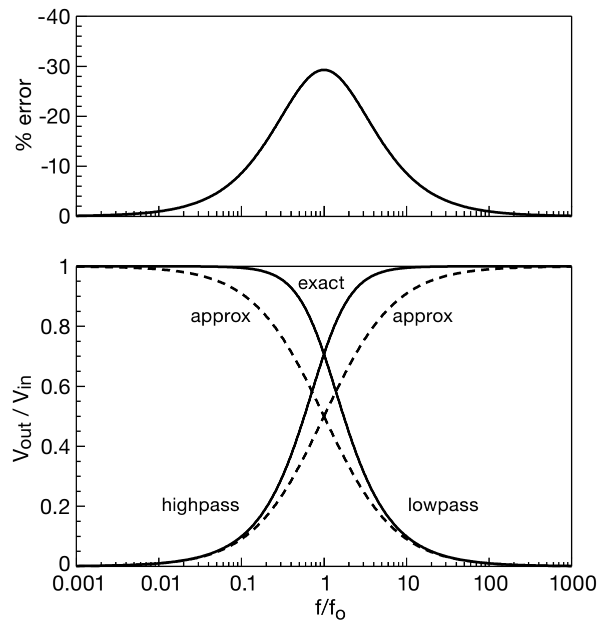


Figure 3: Frequency response of single-section RC filters, showing the results both of a simple approximation that ignores phase (dashed line), and the exact result (solid line). The fractional error (i.e., dashed/solid) is plotted above.

RC High-pass Filter (approximate)

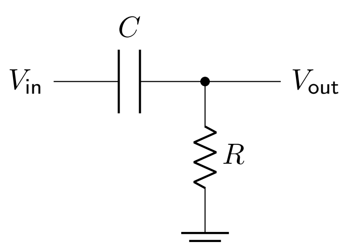


Figure 4: RC high-pass filter.

You get the reverse behavior (pass high frequencies, block low) by interchanging R and C, as in Fig. 4. Treating it as a frequency-dependent voltage divider, and ignoring phase shifts once again, we get (see Fig. 3)

$$\frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{R}{R + X_C} = \frac{R}{R + 1/\omega C} = \frac{\omega RC}{1 + \omega RC} \qquad \text{(approximate!)}$$

High frequencies (above the same crossover frequency as before, $\omega \gg \omega_0 = 1/RC$) pass through (because the capacitor's reactance is much smaller than R), whereas frequencies well below the crossover are blocked (the capacitor's reactance is much larger than R). As before, the equation (and graph) are accurate at both ends, and only modestly in error at the crossover, where the correct ratio is, again, $V_{\text{out}}/V_{\text{in}} = 1/\sqrt{2}$.

RC Filters, Exact Results

Let's do the RC lowpass filter (Fig. 2) in all its complex glory.[2] Here we treat V_{out} and V_{in} as complex numbers, but we still start with our familiar voltage divider equation:

$$V_{out} = V_{in}\frac{Z_C}{Z_R + Z_C} = V_{in}\frac{-j/\omega C}{R - j/\omega C} = V_{in}\frac{1}{1 + j\omega RC}. \tag{13}$$

The transfer function (or "gain") of this filter is

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}. \tag{14}$$

We can compute the magnitude (i.e., multiply by the complex conjugate and take the square root if you need a reminder of the drill) and get

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{(1 + \omega^2 R^2 C^2)^{1/2}}. \tag{15}$$

To compute the complex phase, we first want to make the denominator real, so we multiply the numerator and the denominator by $(1 - j\omega RC)$ to get

$$\frac{V_{out}}{V_{in}} = \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}. \tag{16}$$

The phase angle is thus given by

$$\arg\left(\frac{V_{out}}{V_{in}}\right) = \arg(1 - j\omega RC) = \tan^{-1}\left(\frac{-\omega RC}{1}\right). \tag{17}$$

At the crossover frequency $\omega = 1/RC$ (or $f = 1/(2\pi RC)$), the output amplitude is diminished by

$$\left|\frac{V_{out}}{V_{in}}\right|_{\omega=1/RC} = \frac{1}{\sqrt{2}}. \tag{18}$$

Because $1/\sqrt{2}$ corresponds to -3 dB, the crossover frequency is also called the "3 dB frequency". The complex phase at the crossover frequency is

$$\arg\left(\frac{V_{out}}{V_{in}}\right)_{\omega=1/RC} = \tan^{-1}(-1) = -45°. \tag{19}$$

That is, the output voltage is lagging behind the input voltage by 45 degrees.

For completeness, here is the expression for the RC high-pass filter:

$$V_{out} = V_{in}\frac{R}{R - j/\omega C} = V_{in}\frac{j\omega RC}{1 + j\omega RC}. \tag{20}$$

[2] You won't be expected to do this math by hand; here we are just showing where the result comes from.

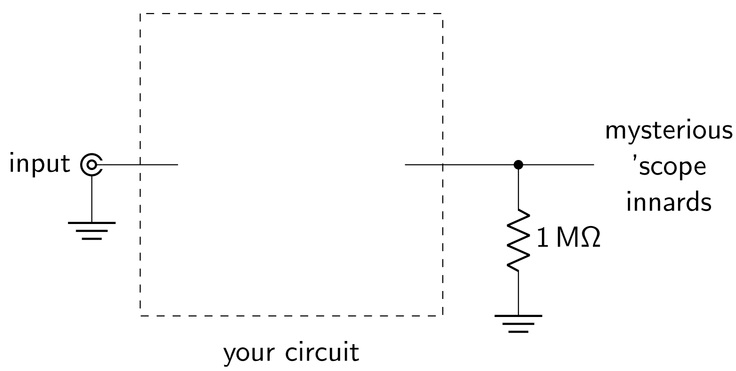
Pre-lab 5 Hand-in

Name: ______________________________

Pre-lab 5.1 The oscilloscope has a mode called "AC Coupling" which places a filter in the path of the signal, and allows you to look only at the oscillating (AC) parts of a signal.

(a) Is the filter low-pass or high-pass?

(b) Given the specifications that the 3 dB frequency of this circuit is 10 Hz, and the downstream input resistance of the scope's guts is 1 MΩ, complete the diagram below.



From the last lab:

Here are some quick questions to help focus your lab report. You should be able to find the answers in the lab, lecture, or prelab from last time.

Pre-lab 5.2 What is the equation of the phase relationship you measured in part 2 of the last lab? What parameter(s) did you vary experimentally and which remained fixed in your measurement?

Pre-lab 5.3 In your report, you should fit the expected relationship (which you wrote down above) to your actual data. It's possible that the fit parameters won't match the theoretical parameters. What kinds of mismatches could you see, and what would they mean physically?