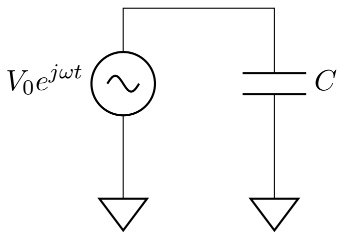
Lecture 4: AC signals in RC circuits

Last time, we looked at RC and CR circuits attached to a constant voltage source. Today we will explore AC signals.

Contents

1 Resistance, Reactance, and Impedance



When we have a sinusoidal input signal $V(t) = V_0 \sin \omega t$, we may want to compare the phase difference between voltage and/or current at different points in the circuit. Let's take one of the simplest examples, of some voltage source connected to a capacitor.

$$j \equiv i = \sqrt{-1}$$

$$V_0 e^{j\omega t} = V_0[\cos \omega t + j \sin \omega t]$$

In this simple case, it's easy to see that the current *leads* the voltage by 90° since they are simple sine and cosine functions. In more complicated circuits, we may not have a simple sine and cosine relationship. In this case, it's easier to view the relative phase if the signals are written in complex exponential form.

$$\begin{aligned} I &= C\frac{dV}{dt} \\ &= C\frac{d}{dt}(V_0 \sin \omega t) \\ &= C\omega V_0 \cos \omega t \end{aligned}$$

$$\begin{aligned} I &= C\frac{dV}{dt} \\ &= C\frac{d}{dt}e^{j\omega t} \\ &= C\omega V_0 j e^{j\omega t} \\ &= C\omega V_0 e^{j(\omega t + \pi/2)} \end{aligned}$$

When we want to find the current through a component, we want to be able to use some rule similar to Ohm's law instead of solving a differential equation every time (which could be complicated). We will introduce the complex impedance Z to accomplish this.

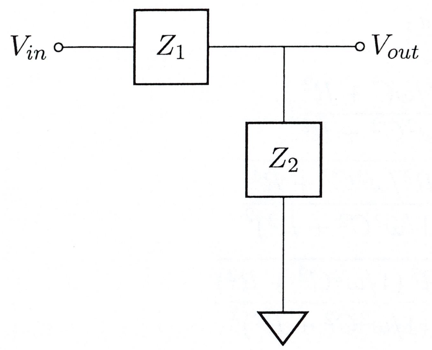
$$Z = R + jX = \frac{V}{I}$$

for the capacitor circuit above, this gives

$$\begin{aligned} &= \frac{V_0 e^{j\omega t}}{C\omega V_0 e^{j(\omega t + \pi/2)}} \\ &= \frac{1}{j\omega C} = \frac{-j}{\omega C} \end{aligned}$$

This value is completely imaginginary, so we can say $Z_C = \frac{-j}{\omega C}$ for a capacitor, or $X_C = \frac{-1}{\omega C}$. At low frequencies, a capacitor acts like an open circuit (infinite impedance). At high frequencies, a capacitor acts like a short circuit (zero impedance).

2 Generalized Voltage Divider, Transfer Function, and Gain



Now we can use our general concept of impedance to find the voltage signal at any point in our circuit, by creating a generalized voltage divider. You already know how the voltage divider will work:

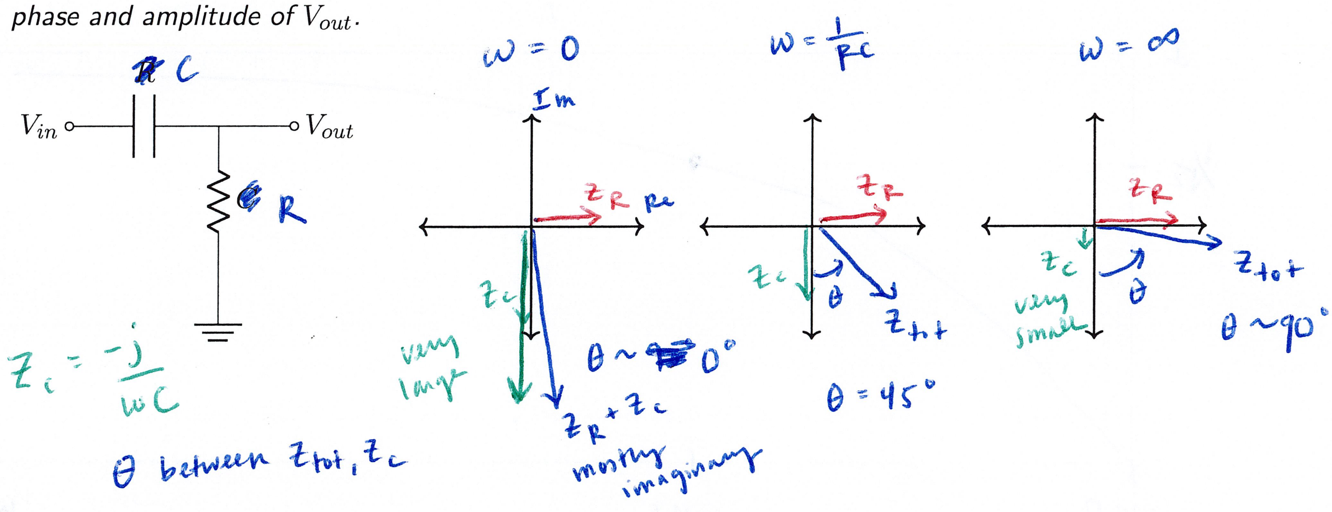
$$V_{out} = V_{in}\frac{Z_2}{Z_1 + Z_2}$$

$\frac{V_{out}}{V_{in}}$ is called the transfer function, and could be complex valued implying a phase shift between V_{out} and V_{in}.

The voltage gain, $G = |\frac{V_{out}}{V_{in}}|$ is real valued and only tells us about the amplitude change.

3 Complex impedance: the whole picture

Now let's worry (briefly) about the fact that Z is complex, and see what implications that has for the phase and amplitude of V_{out}.



To find the phase between V_{in} and V_{out}, we will want to do some phasor analysis. We can imagine complex signals living in the complex plane, with the real component on the horizontal axis and the imaginary component on the vertical axis. That would leave V_{in} entirely on the horizontal axis since we take V_{in} to be our reference signal and have zero phase. Then V_{out} will have some imaginary and real components that leave it oriented with some angle $\phi = \arctan\left(\frac{\text{Im}(V_{out})}{\text{Re}(V_{out})}\right)$.

$$
\begin{aligned}
V_{out} &= V_{in}\frac{Z_2}{Z_1 + Z_2} \\
&= V_{in}\frac{R}{-j/\omega C + R} \\
&= V_{in}\frac{R(j/\omega C + R^2)}{1/\omega^2 C^2 + R^2}
\end{aligned}
$$

$$
\begin{aligned}
\tan\phi &= \frac{\mathrm{Im}(V_{out})}{\mathrm{Re}(V_{out})} \\
&= \frac{R/\omega C}{R^2} = \frac{1}{\omega RC}
\end{aligned}
$$

The phase difference is $\phi = \arctan(1/\omega RC)$.

The gain is

$$
\begin{aligned}
G &= |\frac{V_{out}}{V_{in}}| \\
&= |\frac{jR/\omega C + R^2}{1/\omega^2 C^2 + R^2}| \\
&= \sqrt{\frac{R^2/\omega^2 C^2 + R^4}{(1/\omega^2 C^2 + R^2)^2}} \\
&= \sqrt{\frac{R^2\,(1/\omega^2 C^2 + R^2)}{(1/\omega^2 C^2 + R^2)^2}} \\
&= \frac{\omega C}{\sqrt{1/\omega^2 C^2 + R^2}}
\end{aligned}
$$