Tutorial: Closed Orbits, Energy, and Angular Momentum

In lecture, we've been studying orbital motion under the influence of a $1/r$ potential. Although our results are more general, we're most interested in the case of a small object orbiting a large one under the influence of gravity.

The equation of orbit and orbital speed for this situation can be written:

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi} \qquad (1)$$

$$v(\phi) = \sqrt{\frac{GM}{c}}\sqrt{\frac{c^2}{r^2} + \epsilon^2 \sin^2\phi} \qquad (2)$$

$c = \frac{\ell^2}{\mu\gamma}$

As we've seen, the equation of orbit implies relations between orbital geometry and a few important quantities:

$$E = -\frac{GMm}{2a} \qquad (3)$$

$$L_z = \mu\sqrt{GMc} \qquad (4)$$

$$\tau^2 = \frac{4\pi^2}{GM}a^3 \qquad (5)$$

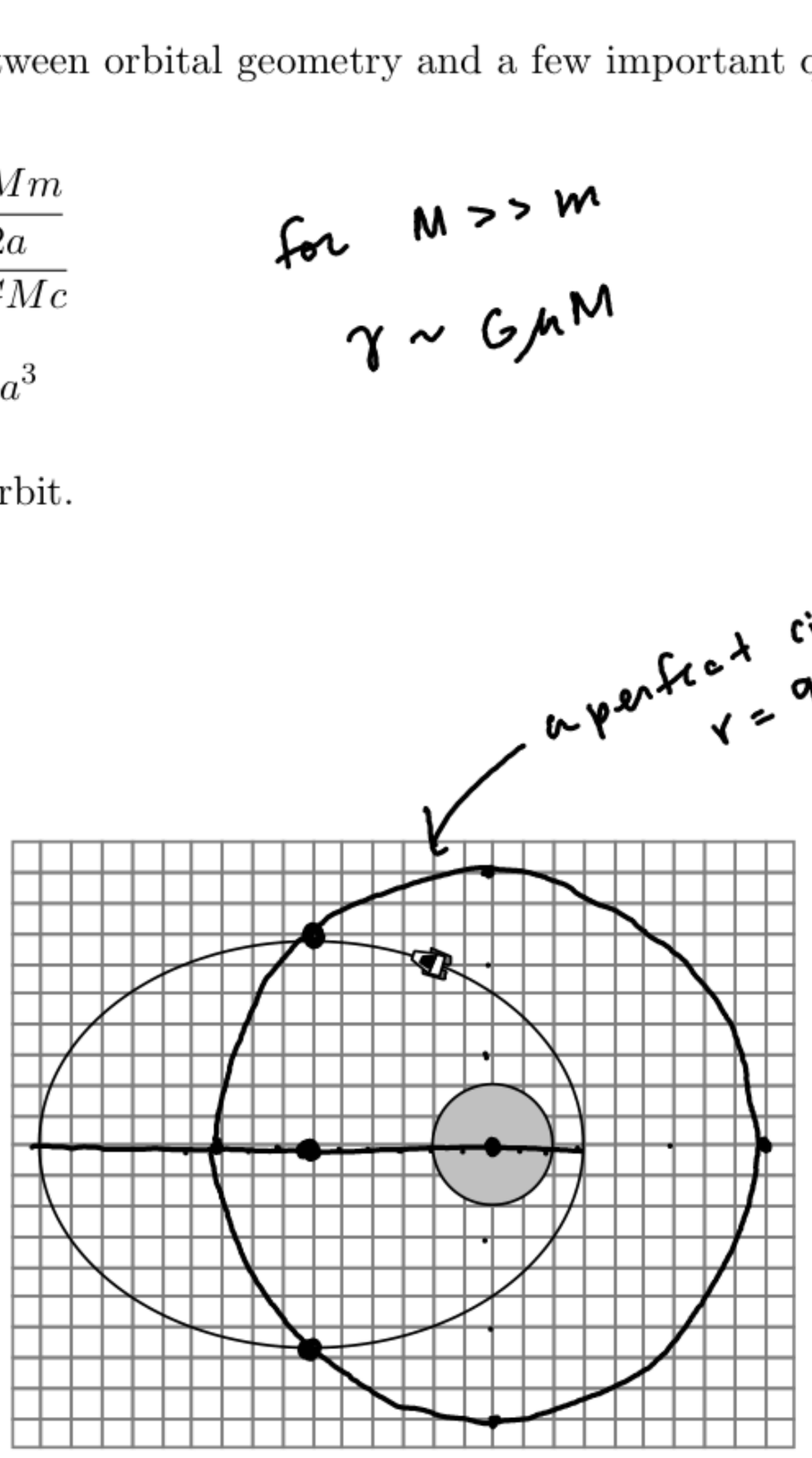
for $M >> m$

$\gamma \sim G/M$

where $a = (r_{\min} + r_{\max})/2$ is the semimajor axis of the orbit.

I. Comparison of circular and elliptical orbits

Imagine that you are piloting a shuttlecraft around an uncharted planet. You bring your shuttle into an elliptical orbit, as shown to the right. The distance from your shuttlecraft to the center of mass (i.e. the center of the planet) varies from a minimum of 3 units to a maximum of 15 units.



A. What is the radius of the *circular orbit* in which the shuttle-planet system would have the same total energy as for the elliptical orbit? Sketch this orbit carefully on the diagram.

total energy only depends on a,
so it's a circle w/ r = a
a = 9 (from counting), centered at CM

Now consider a location where both orbits intersect.

B. In which orbit (if any) would the shuttle pass through that location with the *faster* speed?

r = 9 for both orbits at intersection point

$E = T + U(r)$ same $U(r)$ same E $\longrightarrow$ same T!

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C. In which orbit (if any) would the shuttle pass through the intersection with the *greater* angular momentum? Justify your answer in two different ways:

- using the geometry of the orbits;

$L_z = \mu\sqrt{GMc}$

$a = \frac{c}{1-\varepsilon^2}$ $\quad c = a(1-\varepsilon^2)$

depends on ε!

$c_{ellipse} < c_{circle}$

so L_z greater for circle

- using the definition $L_z = |\vec{r} \times \vec{p}|$ and your answer to part B.

$L_{ellipse} = r m v \sin\varphi$ $\qquad L_{circle} = r m v$

$\cos\varphi \leq 1$ so L_{circle} is greater than $L_{ellipse}$

D. [Discussion] In lecture, we derived the orbital energy E using just the equations for $r(\phi)$ and $v(\phi)$ at $r_{\min}$. Can you do the same derivation for L_z?

STOP HERE and wait to discuss the previous section as a group.
(scratch space below)

$r(\varphi) = \frac{c}{1+\varepsilon\cos\varphi}$ $\qquad v(\varphi) = \sqrt{\frac{GM}{c}}\sqrt{\frac{c^2}{r^2} + \varepsilon^2\sin^2\varphi}$

Since L_z is conserved, take $\varphi = 0$

$L_z = \vec{r} \times m\vec{v} = \frac{c}{1+\varepsilon} \cdot m\sqrt{\frac{GM}{c}}\sqrt{\frac{\varepsilon^2}{\varepsilon^2}(1+\varepsilon)^2 + 0}$

$= m\sqrt{GMc}$ ✓

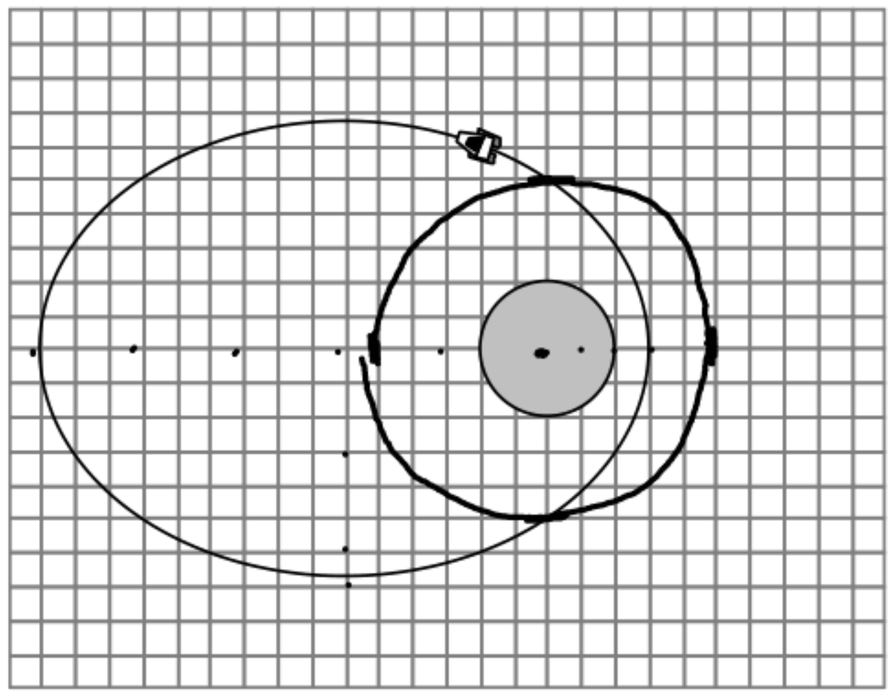
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II. Comparison of orbits, continued

$c = r_{max}(1-\varepsilon) = r_{min}(1+\varepsilon)$

$(r_{max} + r_{min})\varepsilon = r_{max} - r_{min}$

A. Now find the radius of the circular orbit in which the shuttle would instead have the same angular momentum as for the elliptical orbit, and sketch it on the diagram.



$L_z = m\sqrt{GMc}$ need c!

$r_{min} = \frac{c}{1+\varepsilon} = 3$

$r_{max} = \frac{c}{1-\varepsilon} = 15$

$c = 5$

$= r$

B. Consider a location in space where both orbits intersect. In which orbit (if any) would the shuttle pass through that location with the *faster* speed? Again justify your answer two different ways:

- using your knowledge about the angular momentum for each orbit, and

ellipse is faster!

$L_z = \vec{r} \times m\vec{v}$

ellipse: $L_z = rmv\sin\varphi$ circle: $L_z = rmv$

$v = L_z/rm\sin\varphi > L_z/rmv$ b/c $\sin\varphi < 1$

- using your knowledge about the total energy for each orbit.

$E = -GMm/2a = \frac{1}{2}mv^2 + U(r)$

ellipse: $a = 9$

circle: $a = 5$

$\longrightarrow$ circle has less energy (more negative) $\rightarrow$ $T_e > T_c$, $V_e > V_c$

C. What are the eccentricity ϵ and scale factor c of the elliptical orbit?

Found to do part a. $c = 5$

$a = \frac{c}{1-\varepsilon^2} \rightarrow \varepsilon^2 = -\frac{c}{a} + 1$

$= -\frac{5}{15} + 1 = +\frac{2}{3}$

$\varepsilon = \sqrt{2/3}$

D. [Discussion] Which orbit (on this page) has the longer period, the elliptical or circular orbit? If two shuttles start in these orbits from $\phi = 0$, how many times will one shuttle pass the other one before the slower shuttle returns to $\phi = 0$? $\tau^2 \propto a^3 \rightarrow$ ellipse has longer period

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$\frac{T_e}{T_c} = \left(\frac{9}{5}\right)^{3/2} \sim 2.415$

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