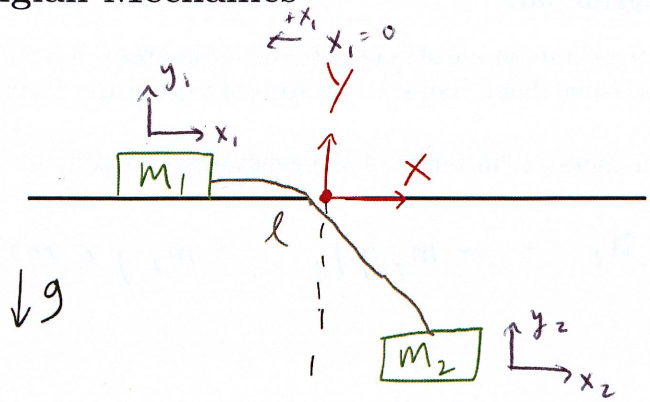
Tutorial: Lagrangian Mechanics



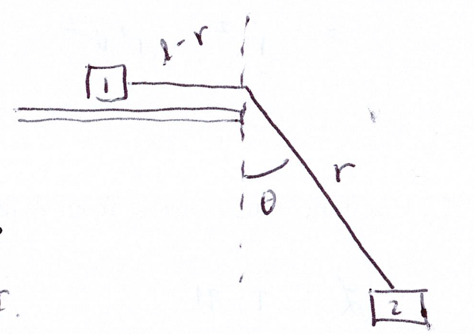
Two masses m_1 and m_2 are connected by a massless rope of length ℓ. As pictured, one of them rests on a frictionless table, while the other is dangling over the edge. We'll treat this as a **two-dimensional** system, starting with Cartesian coordinates (x, y) as depicted. Note that in particular, this means that **block 2 is free to swing back and forth!**

I. Choosing coordinates

A. Using the Cartesian coordinates of the two blocks (x_1, y_1) and (x_2, y_2), write down a set of equations which represent all of the constraints on the system as given.

Block 1 sits on table: $y_1 = 0$

Rope has fixed length: $l^2 = x_1^2 + x_2^2 + y_2^2$



B. How many degrees of freedom does this system have?

4 dim − 2 constraints = 2 d.o.f.

C. Sketch a set of generalized coordinates on the diagram above. $\boxed{r, \theta}$

D. [Discussion] By treating this problem as two-dimensional, we are *ignoring* the coordinates z_1 and z_2, knowing that if $z_1(0) = z_2(0) = 0$, the system doesn't move in the z-direction. What about the coordinate x_2 - is it *ignorable*? Why or why not?

no! free to swing around.

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II. Setting up the Lagrangian

On the board, we've set up a common set of generalized coordinates (GCs) to use, and re-expressed the Cartesian coordinates using them. Let's go on to write down the Lagrangian describing this system.

A. Write the potential energy U in terms of the generalized coordinates.

$$U = \cancel{U_1} + U_2 = -m_2 g y_2 = -m_2 g r \cos\theta$$

B. Evaluate the time derivatives of the Cartesian coordinates in terms of the GCs (and their derivatives), and use your results to write the kinetic energy T.

$$v_1 = \frac{d}{dt}(\ell - r) = -\dot{r}$$

$$v_2^2 = \left(\frac{d}{dt}x_2\right)^2 + \left(\frac{d}{dt}y_2\right)^2$$

$$= \left(\frac{d}{dt} r\sin\theta\right)^2 + \left(\frac{d}{dt} r\cos\theta\right)^2$$

$$= (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2 + (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

$$T = \frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_2\dot{r}^2 + \frac{1}{2}m_2 r^2\dot{\theta}^2$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_2 r^2\dot{\theta}^2$$

C. Combine your answers from A and B to obtain the Lagrangian, $\mathcal{L}$.

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + \frac{1}{2}m_2 r^2\dot{\theta}^2 + m_2 g r\cos\theta$$

D. *[Discussion]* Suppose that instead of having a fixed length, the rope connecting the two blocks is spooled inside the block on the table, and at time $t = 0$ begins to unwind at a constant rate $d\ell/dt = w$. Do we have to modify our GCs to take this into account? How about the potential energy? The kinetic energy?

would change $v_1 \to w - \dot{r}$

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