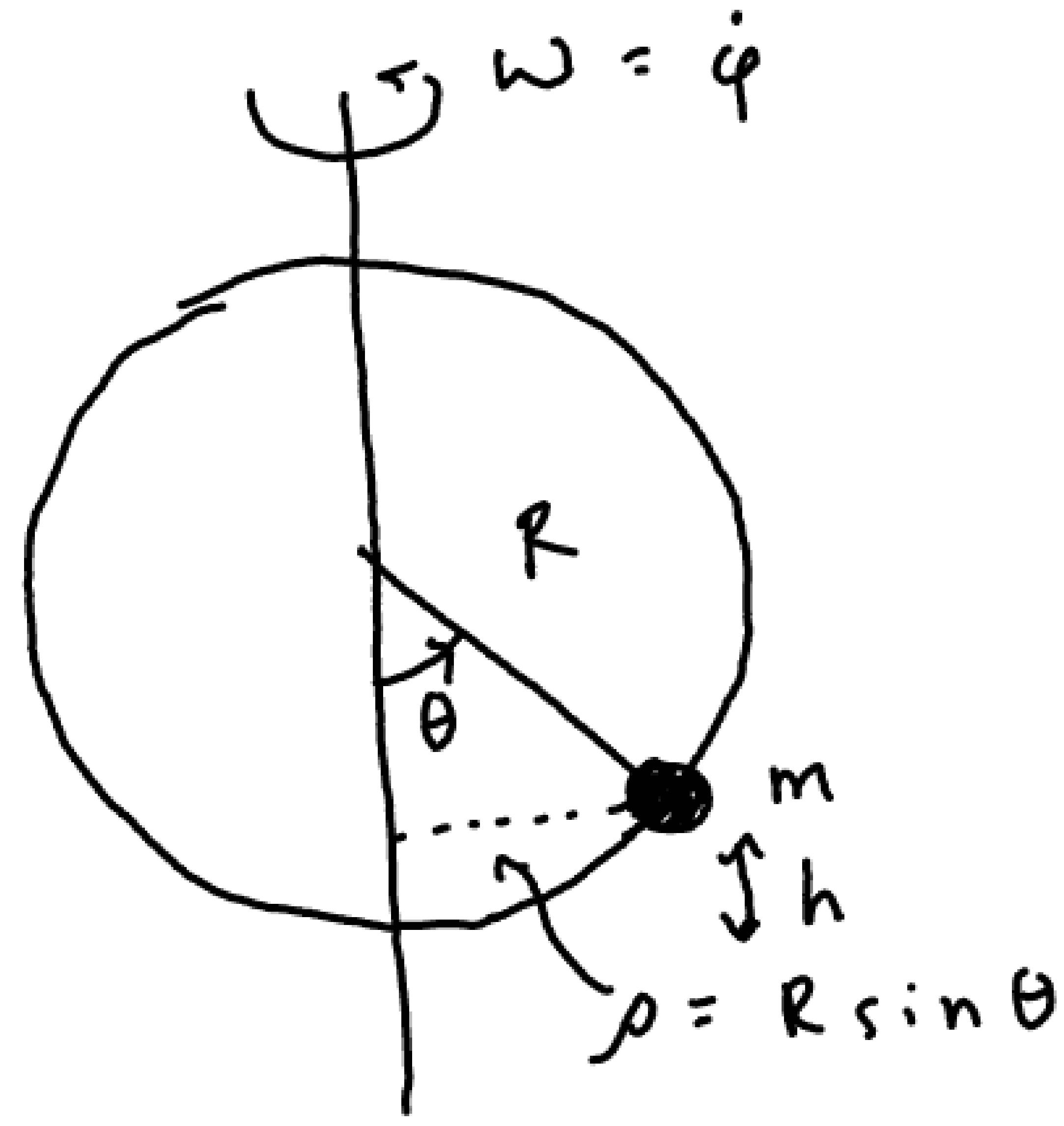


This system is <u>non-relativistic</u> because the hoop is spinning, which gives the bead centripedal acceleration.

First, write the Lagrangian in inertial coordinates. We can fully specify the motion with θ.

motion is purely rotational on two axes: θ, φ

$$T = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m \rho^2 \dot{\varphi}^2$$

$$= \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta \right) \qquad \text{using } \dot{\varphi} = \omega$$

$$\mathcal{U} = mgh = mg(R - R\cos\theta) \qquad \text{ignore constant term.}$$

$$\mathcal{L} = T - \mathcal{U}$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \omega^2 \sin^2\theta + mgR\cos\theta$$

Solve Euler-Lagrange in θ

$$\frac{\partial \mathcal{L}}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2\ddot{\theta}$$

$$\cancel{m}R^{\cancel{2}}\omega^2\sin\theta\cos\theta - \cancel{m}g\cancel{R}\sin\theta = \cancel{m}R^{\cancel{2}}\ddot{\theta}$$

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

$$\ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta$$

which we cannot solve analytically

<u>BUT</u> we can look at behavior in some useful limiting cases.

Equilibrium Positions

Any point where if our object starts there at rest, then it will remain at rest.

In mathematical terms, $\left.\frac{d\dot{\theta}}{dt}\right|_{\dot{\theta}=0} = \left.\ddot{\theta}\right|_{\dot{\theta}=0} = 0$.

Then $\left(w^2 \cos\theta - \frac{g}{R}\right) \sin\theta = 0$

Two eq. points: $\sin\theta = 0 \rightarrow \theta = 0$ or π

bottom or top of hoop.

$w^2 \cos\theta_0 = g/R$

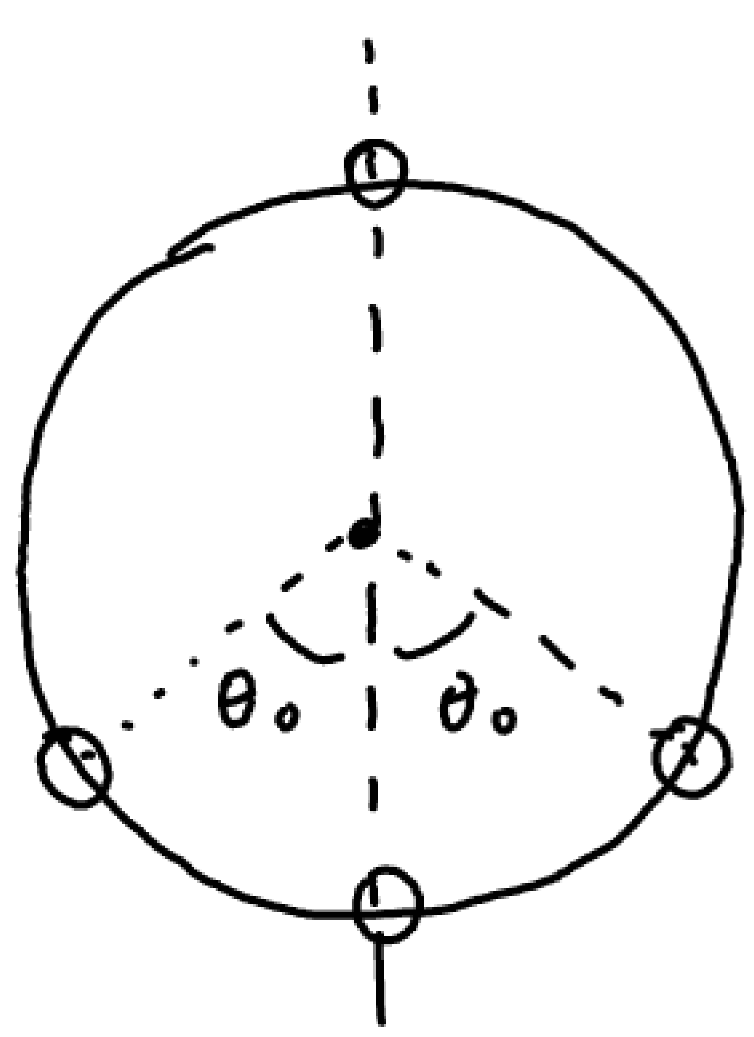
$\cos\theta_0 = \frac{g}{w^2 R}$ $\leftarrow$ $|\cos\theta| \leq 1$

so $\frac{g}{w^2 R} \leq 1$

$\theta_0 = \pm \cos^{-1}\left(\frac{g}{w^2 R}\right)$

$w^2 > g/R$

condition for second set of eq. points.

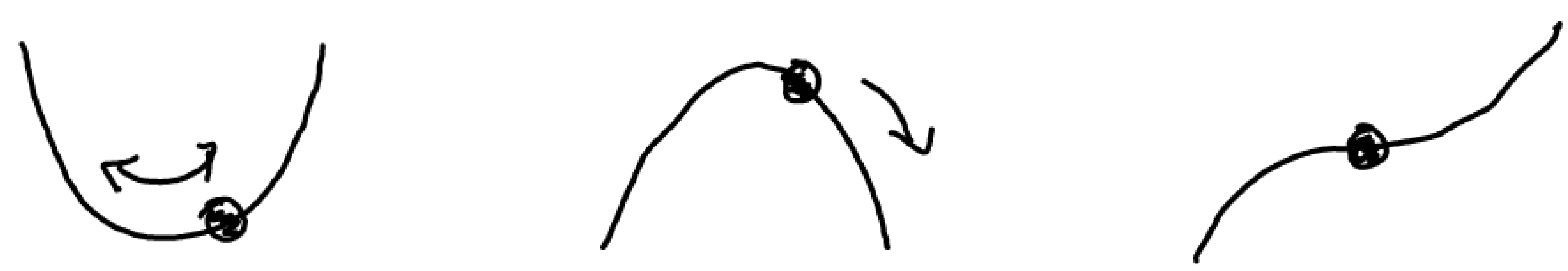


Four possible eq. points for high speed $w^2 > g/R$

Only two for low speed $w^2 < g/R$

Stability

once we have found equilibrium points, we need to know their behavior.



stable unstable semistable

Our general strategy is to take a small expansion around the equilibrium point

$$\theta = \theta_0 + \varepsilon$$ where ε is a function of time and very small.

$\theta_0 = 0$ eq point :

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta$$

$$\ddot{\varepsilon} = \left(\omega^2 \cos(\cancel{\theta_0}^{\,0} + \varepsilon) - \frac{g}{R}\right)\sin(\cancel{\theta_0}^{\,0} + \varepsilon)$$

for small ε,

$\cos\varepsilon \sim 1$ $\quad$ $\sin\varepsilon \sim \varepsilon$

$$\ddot{\varepsilon} = (\omega^2 - g/R)\,\varepsilon$$

$\ddot{\varepsilon} = (\omega^2 - g/R)\,\varepsilon$ is a familiar form

If $\omega^2 - g/R > 0$ then the solution is exponential.

$$\varepsilon(t) = A e^{ct} + B e^{-ct} \quad \text{where } c = \sqrt{\omega^2 - g/R}$$

you can fill in initial conditions to find the exact form, but the result will always be unstable.

If $\omega^2 - g/R < 0$ then we have a SHO

$$\varepsilon(t) = A e^{ict} + B e^{-ict}$$

this is stable, oscillating behavior.

Repeating for $\theta_0 = \pi$:

$$\ddot{\theta} = (\omega^2 \cos\theta - g/R) \sin\theta$$

$$\ddot{\varepsilon} = (\omega^2 \cos(\pi + \varepsilon) - g/R) \sin(\pi + \varepsilon)$$

$\cos(\pi + \varepsilon)$ will always be close to -1

I'll expand $\sin(\pi + \varepsilon)$

as $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$$\sin(\pi + \varepsilon) = \cancel{\sin\pi}^{\,0} \cos\varepsilon + \cos\pi \sin\varepsilon$$

$$\sim -\sin\varepsilon$$

for small ε, $\sin\varepsilon \sim \varepsilon$

$$\ddot{\varepsilon} = (+\omega^2 + g/R)\,\varepsilon \qquad \omega^2 + g/R > 0$$

so this point is unstable

ω - dependent eq. points

$$0 < \theta_0 < \pi/2$$

$$\ddot{\theta} = \underbrace{(\omega^2 \cos\theta - g/R)}_{\text{zero at } \theta_0} \underbrace{\sin\theta}_{\text{positive at } \theta_0}$$

positive for $\uparrow \theta$
negative for $\downarrow \theta$

zero at θ_0

negative if we $\uparrow \theta$
positive if we $\downarrow \theta$

positive at θ_0
no change in sign
w/ changing θ

If we $\uparrow \theta$ from θ_0

OR

$\downarrow \theta$ from θ_0

both have SHO form

$$\ddot{\varepsilon} = -c^2 \varepsilon$$

when they exist $(\omega^2 > g/R)$

they are stable equilibrium points