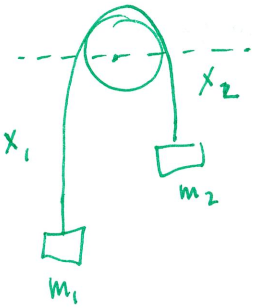
Atwood Machine



constraint: $x_1 + x_2 + \pi R = l$

$$x_1 = -x_2 + \text{const.}$$

$$\dot{x}_1 = -\dot{x}_2$$

reduce to 1 variable w/ constraint

$$\cancel{T = \frac{1}{2} m (v_1^2 + v_2^2) = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) = 0}$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{x}_2^2$$

$$\dot{x}_1^2 = \dot{x}_2^2$$

$$\mathcal{U} = -m_1 g x_1 + m_2 g x_2 = \cancel{m_1 g x_1 - m}$$

$$- m_1 g (-x_2 + \text{const.}) - m_2 g x_2$$

$$\mathcal{U} = g x_2 (m_1 - m_2) + \text{const.}$$

$$\mathcal{L} = T - \mathcal{U} = \frac{1}{2} (m_1 + m_2) \dot{x}_2^2 - g x_2 (m_1 - m_2) + \cancel{\text{const.}}$$ ignore

$$\frac{\partial \mathcal{L}}{\partial x_2} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right) = 0$$

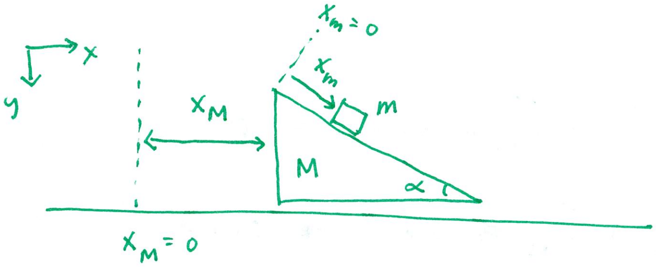
$$-g (m_1 - m_2) - \frac{d}{dt} \left[\cancel{2} (m_1 + m_2) \dot{x}_2 \right] = 0$$

$$(m_2 - m_1) g = (m_1 + m_2) \ddot{x}_2$$

$$\ddot{x}_2 = \left(\frac{m_2 - m_1}{m_2 + m_1} \right) g$$

Sliding block on a sliding ramp

How long does the block take to get to the bottom?



$$T_M = \frac{1}{2} M \dot{x}_M^{\,2}$$

for the block, speed $\dot{x}_m$ is relative to the ramp.

We need to write $\mathcal{L}$ in an inertial frame even if there are relative / accelerating parts in the system.

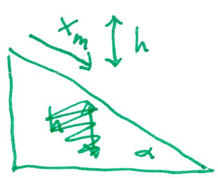
$$\vec{v}_m = (v_{m,x},\ v_{my})$$

$$= (\dot{x}_M + \dot{x}_m \cos\alpha,\ \dot{x}_m \sin\alpha)$$

$$T_m = \frac{1}{2} m (v_x^2 + v_y^2) = \frac{1}{2} m \left(\dot{x}_M^{\,2} + \dot{x}_m^{\,2} \cos^2\alpha + 2\dot{x}_M \dot{x}_m \cos\alpha + \dot{x}_m^{\,2} \sin^2\alpha \right)$$

$$T_m = \frac{1}{2} m \left(\dot{x}_M^{\,2} + \dot{x}_m^{\,2} + 2\dot{x}_M \dot{x}_m \cos\alpha \right)$$

$$T = T_M + T_m = \frac{1}{2}(M+m)\dot{x}_M^{\,2} + \frac{1}{2} m \left(\dot{x}_m^{\,2} + 2\dot{x}_M \dot{x}_m \cos\alpha \right)$$



$$h = -x_m \sin\alpha \qquad \mathcal{U} = -m g x_m \sin\alpha$$

$$\mathcal{L} = T - \mathcal{U}$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}_M^{\,2} + \frac{1}{2} m \left(\dot{x}_m^{\,2} + 2\dot{x}_M \dot{x}_m \cos\alpha \right) + m g x_m \sin\alpha$$

$$\frac{\partial \mathcal{L}}{\partial x_m} = mg\sin\alpha \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_m} = \frac{d}{dt}\left[m\dot{x}_m + m\,\dot{x}_M \cos\alpha \right]$$

$$\frac{\partial \mathcal{L}}{\partial x_m} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_m} = mg\sin\alpha - m\ddot{x}_m - \ddot{x}_M\, m\cos\alpha = 0$$

$$① \quad \ddot{x}_m + \ddot{x}_M \cos\alpha = g\sin\alpha$$

$$\frac{\partial \mathcal{L}}{\partial x_M} = 0 \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_M} = \frac{d}{dt}\left[(M+m)\dot{x}_M + m\dot{x}_m\cos\alpha \right] = 0$$

$$(M+m)\ddot{x}_M + m\ddot{x}_m\cos\alpha = 0$$

$$② \quad \ddot{x}_M = -\frac{m}{M+m}\ddot{x}_m\cos\alpha$$

use ② in ① :

$$\ddot{x}_m - \frac{m}{M+m}\ddot{x}_m\cos^2\alpha = g\sin\alpha$$

$$\ddot{x}_m\left(1 - \frac{m\cos^2\alpha}{M+m}\right) = g\sin\alpha$$

$$\ddot{x}_m = \frac{g\sin\alpha}{1 - \frac{m\cos\alpha}{M+m}}$$

If the ~~block~~ ramp has length l, then the kinematic equation will be $l = \frac{1}{2}\ddot{x}_m t^2$

$$t = \sqrt{2l/\ddot{x}_m}$$

sanity checks :

$\alpha = 90°$, $\dot{x}_m = g$? yes!

$M \to \infty$ $\quad \ddot{x}_M = 0$, $\dot{x}_m = g\sin\alpha$? yes!