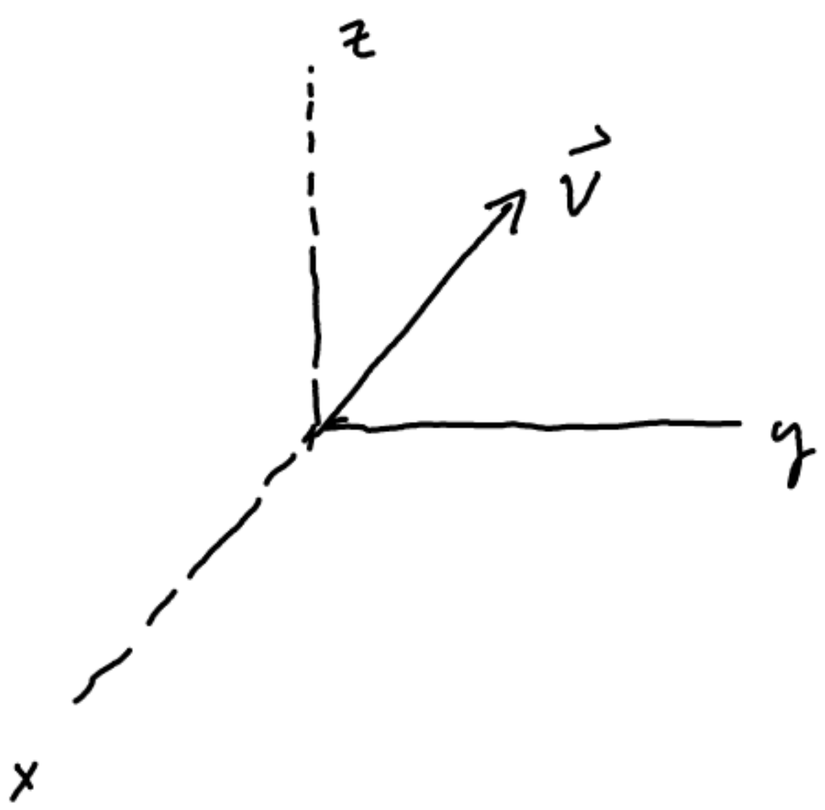
When $\vec{\tau} \neq 0$, we need to write it in the

body frame to use Euler's equations.

Rotation Matrices



We can rotate $\vec{v}$ into any new frame by rotating around the x, y, z axes by some amount

Rules for rotation:

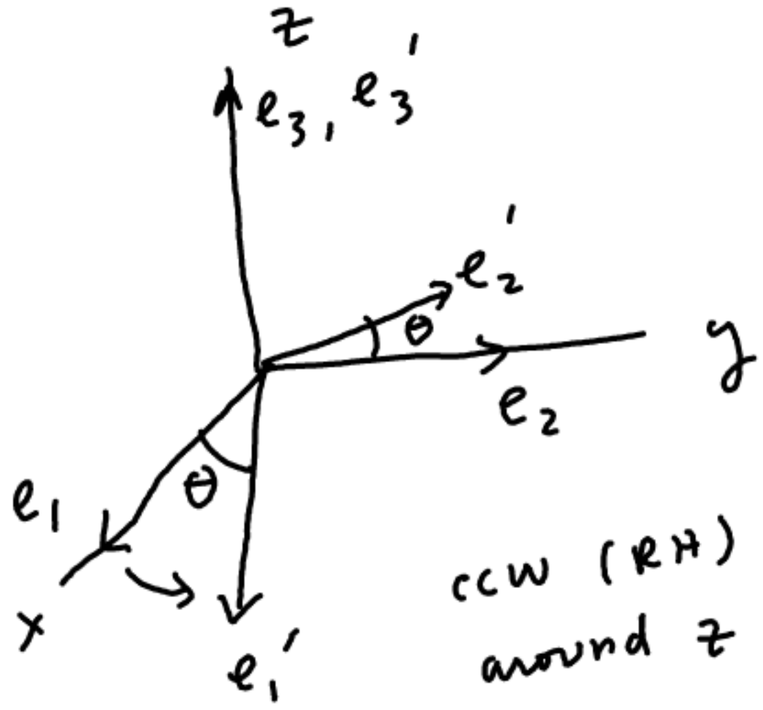
- length doesn't change
- relative angles between vectors don't change
- "orientation" doesn't change ($\alpha \not\to -\alpha$)

↓

We can represent these with matrices

Each rule has a corresponding matrix property

ex: rotation by θ in x-y plane about z-axis



ccw (RH) around z

How to build rotation matrix:

take colums of $\{\vec{e}_i\}$ and rotate by angle θ

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\theta \text{ about } z} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$e_1 \quad e_2 \quad e_3$

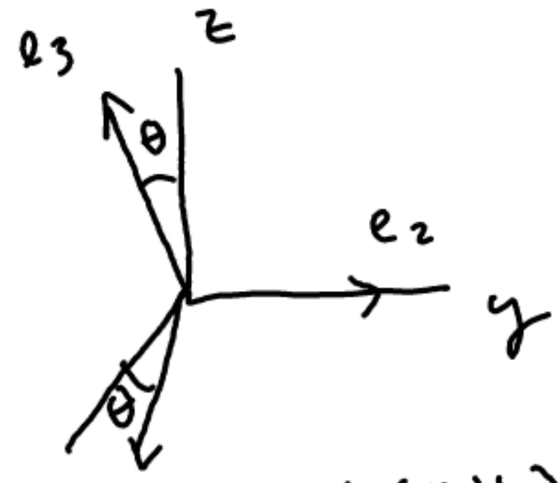
I $\qquad$ $R_z(\theta) \cdot I$

$\Downarrow$

$R_z(\theta)$

$R_z(\theta)\,\vec{v}$ will rotate $\vec{v}$ by θ about z.

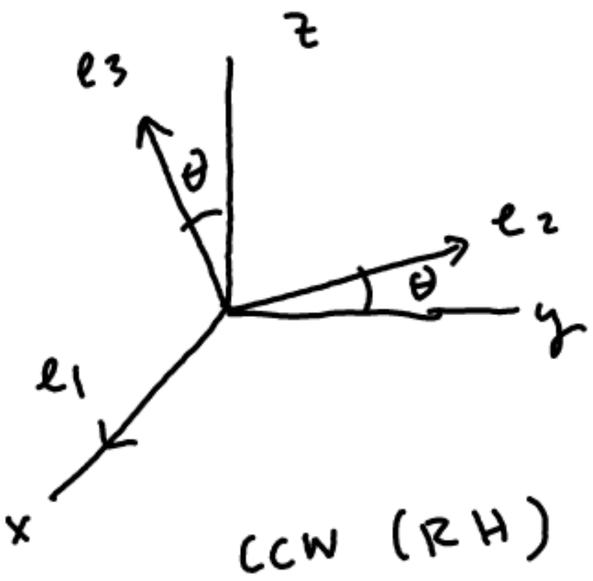
ex: rotate θ in x-z about y



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & & \cos\theta \end{pmatrix}$$

$R_y(\theta)$

ex: rotate θ in y-z about x



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$R_x(\theta)$

I have defined all of my rotations as

RIGHT HANDED.

How would the R-matrices change

for left-handed rotation?

In general, $R = \left[R(\vec{e}_1),\ R(\vec{e}_2),\ R(\vec{e}_3) \right]$

Checking the rules:

① Preserves length

To show vector length is 1, $\vec{v} \cdot \vec{v} = 1$

$\vec{v}^T \vec{v} = 1$

similar property for matrices:

If $R^T R = 1$ then R preserves length

Check for R_z:

$$R_z^+ R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

R_z preserves length

This works generally b/c R-matrices are

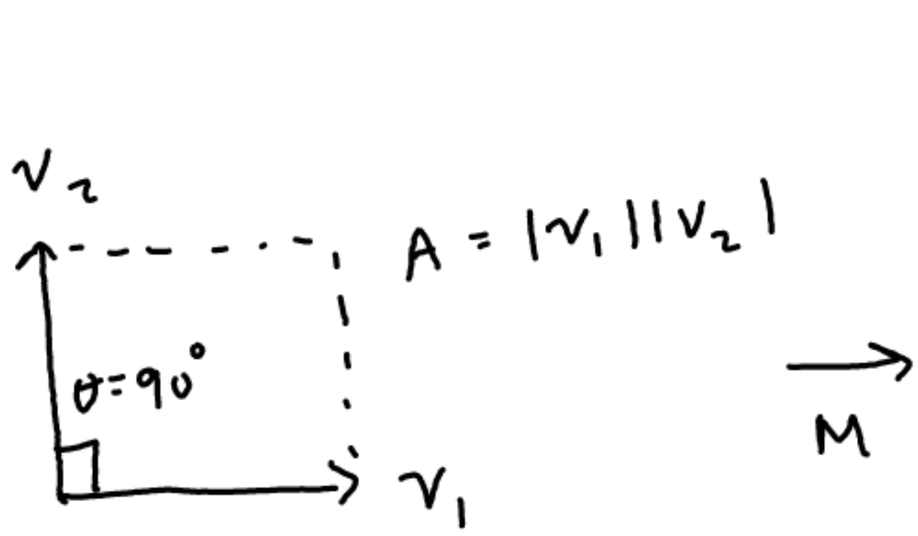
orthonormal $\rightarrow$ $R_i^T \cdot R_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

consequence of preserving both angle + length

Rewrite rule 1 as $R^T R = I$

② Preserve angles

The determinant says something about the angle subtended by two vectors $\vec{v}_1, \vec{v}_2$



$A' = |v_1'||v_2'| \cos\theta$

$= (\det M) \cdot A$

If we preserve angles, A doesn't change

$\downarrow$

$\det M = \pm 1$

so to preserve angles we must have

$\det R = \pm 1$

confirm this:

$$R^T R = I$$

$$\det(R^T R) = \det(I)$$

$$\det(R^T)\det(R) = 1$$

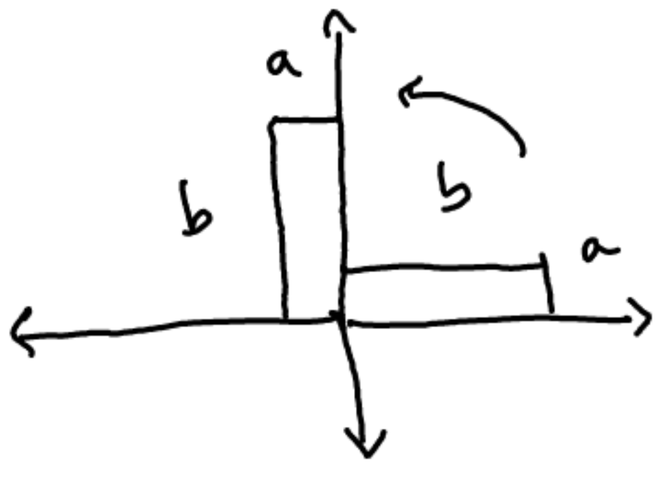
$$\det R^T = \det R$$

$$\det(R)\det(R) = 1$$

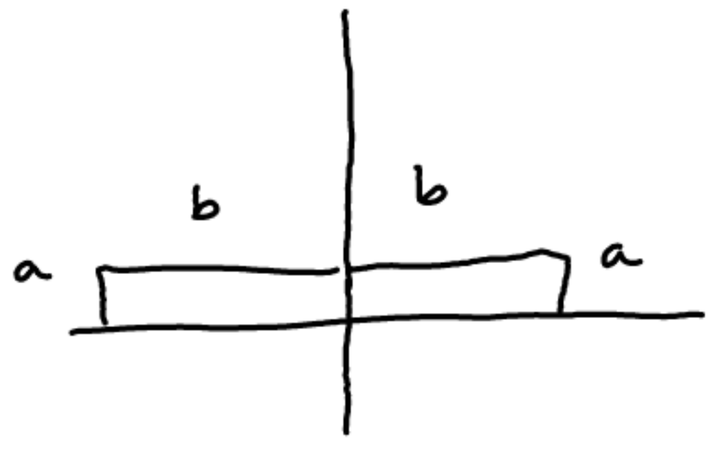
$$\det R = \pm 1$$

(3) orientation does not change

$\det R = +1$ $\qquad$ $\det R = -1$



rotation



reflection

so, R is a rotation IFF: $\det R = +1$

and $R^T R = I$

Composition of Rotations:

Let A, B be rotation matrices

↳ $\det A = 1, \quad A^T A = I$

$\det B = 1, \quad B^T B = I$

$$\det(AB) = \det(A)\det(B) = 1 \quad \checkmark$$

$$(AB)^T(AB) = B^T A^T AB$$

$$= B^T B$$

$$= I \quad \checkmark$$

any composition of rotations is also a rotation.