Equations of Motion for rotating bodies

It is easy to find important quantities like $\vec{L}$ and T in the principle axes:

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega} = (\lambda_1\omega_1,\ \lambda_2\omega_2,\ \lambda_3\omega_3)$$

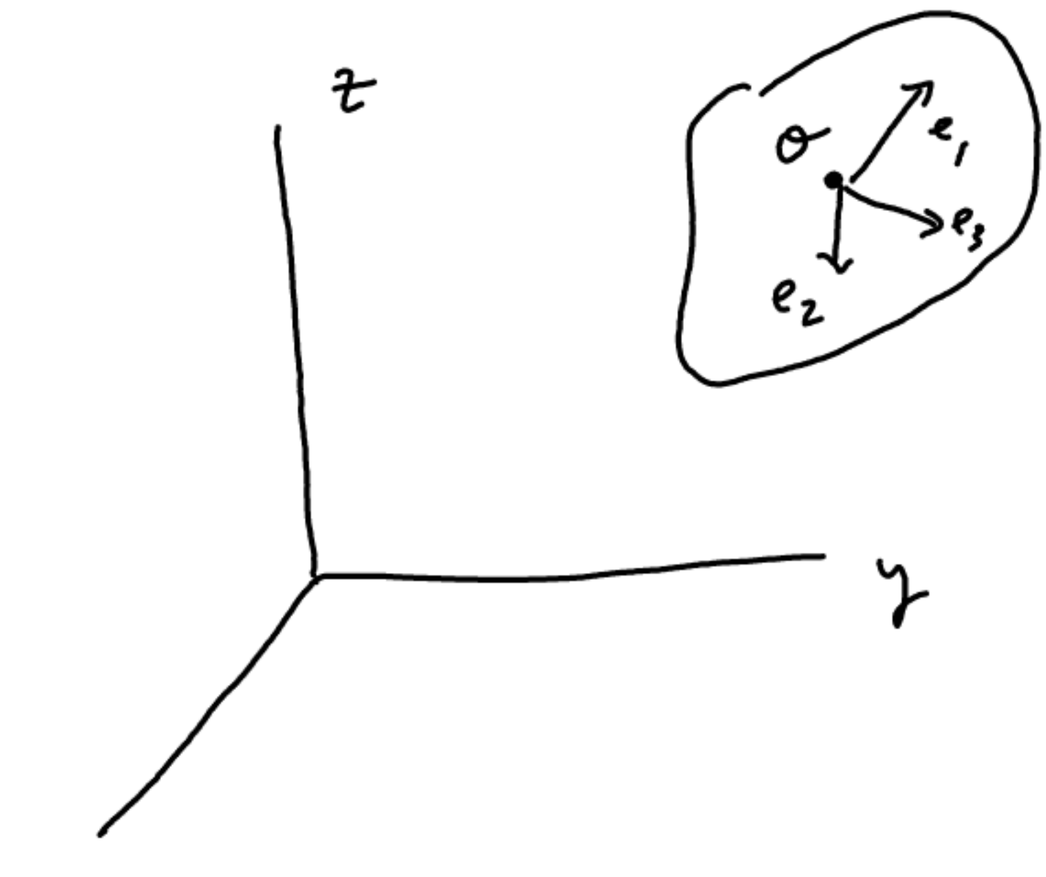
for principle axis $\vec{e}$

$$T = \frac{1}{2}\,\vec{\omega}\cdot\vec{L}$$ ← an interesting exercise to prove

$$= \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right)$$

for principle axis $\vec{e}$

But the principle axes $\{\vec{e}\}$ are attached to the body and rotate in space



space frame

x, y, z

body frame

e_1, e_2, e_3

Torque is acting on the body and should be defined in the space frame:

$$\vec{L} = (w_1\lambda_1,\ w_2\lambda_2,\ w_3\lambda_3) \quad \text{body frame}$$

$$\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{space}$$

same trick from rotating frames

$$= \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

$$\boxed{\vec{\Gamma} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}}$$

Three components:

$$\left.\begin{aligned} \lambda_1 \dot{w}_1 - (\lambda_2 - \lambda_3) w_2 w_3 &= \Gamma_1 \\ \lambda_2 \dot{w}_2 - (\lambda_3 - \lambda_1) w_3 w_1 &= \Gamma_2 \\ \lambda_3 \dot{w}_3 - (\lambda_1 - \lambda_2) w_1 w_2 &= \Gamma_3 \end{aligned}\right\} \text{Euler's eqns}$$

Euler's eqns give evolution of $\vec{\omega}$ as seen from the body frame.

Used for simple cases of torque

($\vec{\tau} = 0$, $\tau_i = 0$, high symmetry)

Special Symmetry Cases

Specify the shape + its pivot point

- spherical top $\quad \lambda_1 = \lambda_2 = \lambda_3$

  all principle axes are degenerate

  It doesn't matter what vectors we choose, as long as they are orthonormal

ex: a sphere about its CM



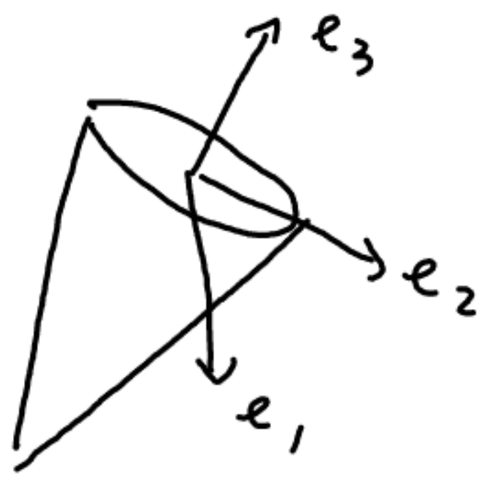
ex: cube about its CM

(prove it?)

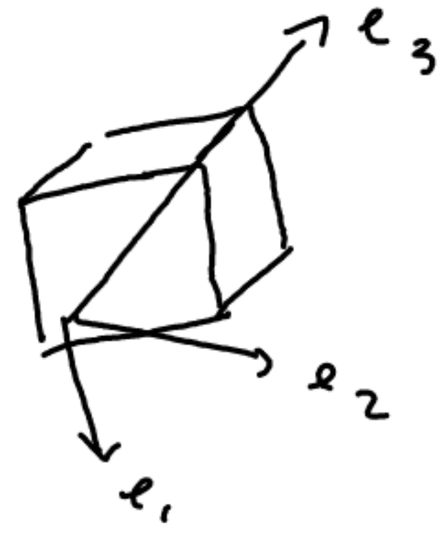
- symmetric top $\lambda_1 = \lambda_2 \neq \lambda_3$

  anything with cylindrical symmetry

ex:



ex: cube about corner



- Linear rotor $\lambda_1 = \lambda_2 >> \lambda_3$

  special symmetric top

  long + narrow shapes (diatomic molecule)

- Asymmetric top $\lambda_1 \neq \lambda_2 \neq \lambda_3$

  most complex

  can use Euler eqns for $\vec{\tau} = 0$

Torque-Free Asymmetric Top

If $\vec{\omega} = \vec{e}_3 \omega_3$ (all along a principle axis)

Then from Euler's eqns $\dot{\omega}_1 = \dot{\omega}_2 = 0$

$\dot{\omega}_3 = 0$

no change in $\vec{\omega}$

true in all frames b/c $\vec{L} = \lambda_3 \vec{\omega}_3$

constant in all frames

If $\vec{\omega}$ is a combination of $\{\vec{e}_i\}$

(not on a principle axis)

then $\dot{\vec{\omega}} \neq 0$

direction of rotation will change

Stability: suppose at $t = 0$ $\omega_1 = \omega_2 = 0$

$\omega_3 \neq 0$

body will rotate around $\vec{e}_3$ only

to test stability, consider a <u>small</u> kick

$\omega_1 \neq 0$

$\omega_2 \neq 0$

$\lambda_3 \dot{w}_3 \sim 0$ b/c $w_1 w_2$ very small

$w_3 \sim$ const.

$$\lambda_1 \dot{w}_1 = [(\lambda_2 - \lambda_3) w_3] w_2$$

$$\lambda_2 \dot{w}_2 = [(\lambda_3 - \lambda_1) w_3] w_1$$

$$\lambda \ddot{w}_1 = [(\lambda_2 - \lambda_3) w_3] \dot{w}_2$$

$$\lambda_1 \ddot{w}_1 = [(\lambda_2 - \lambda_3) w_3] \left[(\lambda_3 - \lambda_1) \frac{w_3}{\lambda_2}\right] w_1$$

Ω

If $\Omega < 0$ then small oscillations ← STABLE

$(\lambda_3 > \lambda_1$ and λ_2 or $\lambda_3 < \lambda_1$ and $\lambda_2)$

If $\Omega > 0$ then exponential ← unstable

$(\lambda_1 < \lambda_3 < \lambda_2$ or $\lambda_2 < \lambda_3 < \lambda_1)$

rotation around largest moment is stable