Eigenvalues + Diagonalized Matrices

Let the matrix M be some transformation on 3-d vectors in space.

examples:
- shrinking or enlarging
- affine deformation
- rotation, reflection

For almost every matrix like this, there is some set of special vectors that are NOT CHANGED by M except by some scalar mult.

EIGENVECTORS and EIGENVALUES

$$M\vec{e} = \lambda\vec{e}$$

for a 3-d M, there will be three $\vec{e}_i$ (orthonormal eigenbasis) with corresponding λ_i.

How to find the eigenvectors:

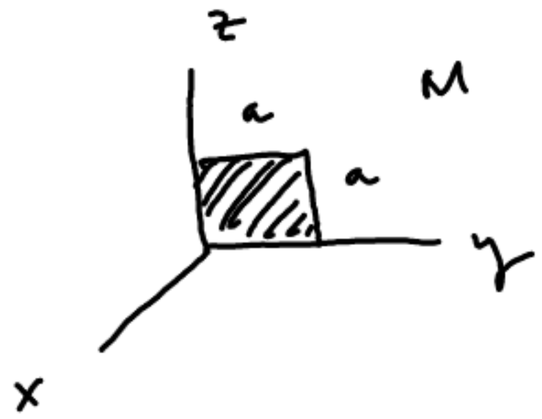
(1) Find eigenvalues

(2) Find corresponding eigenvectors

Example from class: we want to find eigenvectors of $\underline{\underline{I}}$ because we want $\vec{\omega}$ s.t. $\vec{L} \parallel \vec{\omega}$

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega} = \lambda\vec{\omega}$$

For our problem with the rotating square, we found



$$\underline{\underline{I}} = Ma^2 \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & -1/4 \\ 0 & -1/4 & 1/3 \end{pmatrix}$$

we can neglect the constant Ma^2.

(multiply it back in later)

① Find eigenvalues

$$M\vec{e} = \lambda\vec{e}$$

let $\vec{e}$ be some vector $\vec{e} = (x, y, z)$

$$\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & -1/4 \\ 0 & -1/4 & 1/3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

subtract $\lambda\vec{e}$ from both sides

$$\begin{pmatrix} 2/3 - \lambda & 0 & 0 \\ 0 & 1/3 - \lambda & -1/4 \\ 0 & -1/4 & 1/3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

↑

$M - \underline{1}\lambda$

(identity matrix) $\cdot \lambda$

solve this by finding $\det(M - \underline{1}\lambda) = 0$

Boas ch. 3.2

$$\begin{vmatrix} 2/3-\lambda & 0 & 0 \\ 0 & 1/3-\lambda & -1/4 \\ 0 & -1/4 & 1/3-\lambda \end{vmatrix} = 0$$

$$(2/3-\lambda)\left[(1/3-\lambda)(1/3-\lambda) - 1/16 \right] = 0$$

Three solutions, $\lambda_1, \lambda_2, \lambda_3$.

$\lambda_1 = 2/3$

$\lambda_{2,3} \longrightarrow \quad (1/3-\lambda) = \pm \frac{1}{4}$

$$\lambda_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\lambda_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

So the full eigenvalues are (add back in Ma^2)

$$\lambda_1 = \frac{2}{3} Ma^2$$

$$\lambda_2 = \frac{7}{12} Ma^2$$

$$\lambda_3 = \frac{1}{12} Ma^2$$

② Find the eigenvectors for each λ:

1) $M\vec{e}_1 = \lambda_1 \vec{e}_1$

$$\begin{pmatrix} \frac{2}{3}-\frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3}-\frac{2}{3} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{3}-\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/3 & -1/4 \\ 0 & -1/4 & -1/3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0$$

x_1 is undefined

$y_1 = z_1 = 0 \longrightarrow \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

2) $M\vec{e}_2 = \lambda \vec{e}_2$

$$\begin{pmatrix} \frac{2}{3}-\frac{7}{12} & 0 & 0 \\ 0 & \frac{1}{3}-\frac{7}{12} & -1/4 \\ 0 & -1/4 & 1/3-\frac{7}{12} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = 0$$

$x_2 = 0$

$\frac{1}{3}-\frac{7}{12} = \frac{4}{12}-\frac{7}{12} = -\frac{3}{12} = -\frac{1}{4} \longrightarrow y_2 = z_2$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

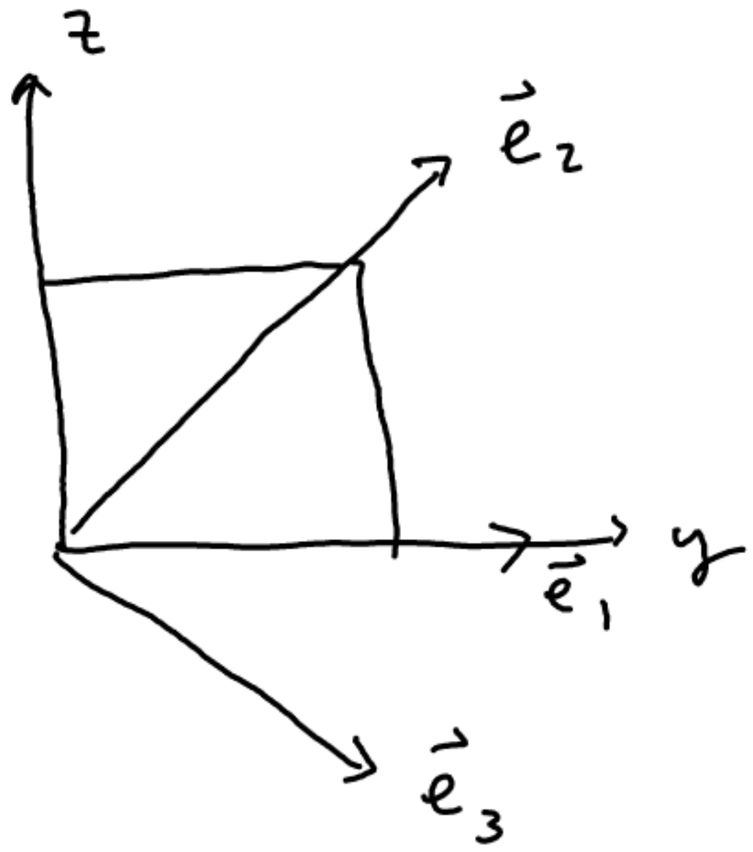
3) $\quad M\vec{e}_3 = \lambda_3 \vec{e}_3$

$$\begin{pmatrix} 2/3 - 1/12 & 0 & 0 \\ 0 & 1/3 - 1/12 & -1/4 \\ 0 & -1/4 & 1/3 - 1/12 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = 0$$

$$x_3 = 0$$

$$\frac{1}{3} - \frac{1}{12} = +\frac{1}{4} \longrightarrow y_3 = -z_3$$

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$



eigenbasis

OR

PRINCIPLE AXES of ROTATION

$$\vec{L}_1 = \lambda_1 \vec{e}_1$$

$$\vec{L}_2 = \lambda_2 \vec{e}_2$$

$$\vec{L}_3 = \lambda_3 \vec{e}_3$$