Keep generalizing! Now let's take rotation around

<u>any</u> arbitrary axis s.t.

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

$$\vec{\omega} \times \vec{r}_\alpha = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = \begin{pmatrix} \omega_y z_\alpha - \omega_z y_\alpha \\ \omega_z x_\alpha - \omega_x z_\alpha \\ \omega_x y_\alpha - \omega_y x_\alpha \end{pmatrix}$$

$$\vec{L}_\alpha = \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha = \vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha)$$

$$\vec{L}_\alpha = \begin{vmatrix} i & j & k \\ x_\alpha & y_\alpha & z_\alpha \\ \omega_y z_\alpha - \omega_z y_\alpha & \omega_z x_\alpha - \omega_x z_\alpha & \omega_x y_\alpha - \omega_y x_\alpha \end{vmatrix}$$

$$= \begin{pmatrix} \omega_x (y_\alpha^2 + z_\alpha^2) - \omega_y y_\alpha x_\alpha - \omega_z z_\alpha x_\alpha \\ \omega_y (x_\alpha^2 + z_\alpha^2) - \omega_x x_\alpha y_\alpha - \omega_z y_\alpha z_\alpha \\ \omega_z (x_\alpha^2 + y_\alpha^2) - \omega_x x_\alpha z_\alpha - \omega_y z_\alpha y_\alpha \end{pmatrix}$$

$$L_x = w_x (y_\alpha^2 + z_\alpha^2) - w_y y_\alpha x_\alpha - w_z z_\alpha x_\alpha$$

$$L_y = w_y (x_\alpha^2 + z_\alpha^2) - w_x x_\alpha y_\alpha - w_z y_\alpha z_\alpha$$

$$L_z = w_z (x_\alpha^2 + y_\alpha^2) - w_x x_\alpha z_\alpha - w_y z_\alpha y_\alpha$$

use $I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2)$ and $I_{xy} = -\sum m_\alpha x_\alpha y_\alpha$

$$L_x = w_x I_{xx} + w_y I_{xy} + w_z I_{xz}$$

$$L_y = w_y I_{yy} + w_x I_{yx} + w_z I_{yz}$$

$$L_z = w_z I_{zz} + w_x I_{zx} + w_y I_{zy}$$

Two ways to summarize this:

1) $$L_i = \sum_{j=1}^{3} I_{ij} w_j$$

2) $$\underline{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

inertia tensor

* notice that

$$\underline{I} = \underline{I}^T$$

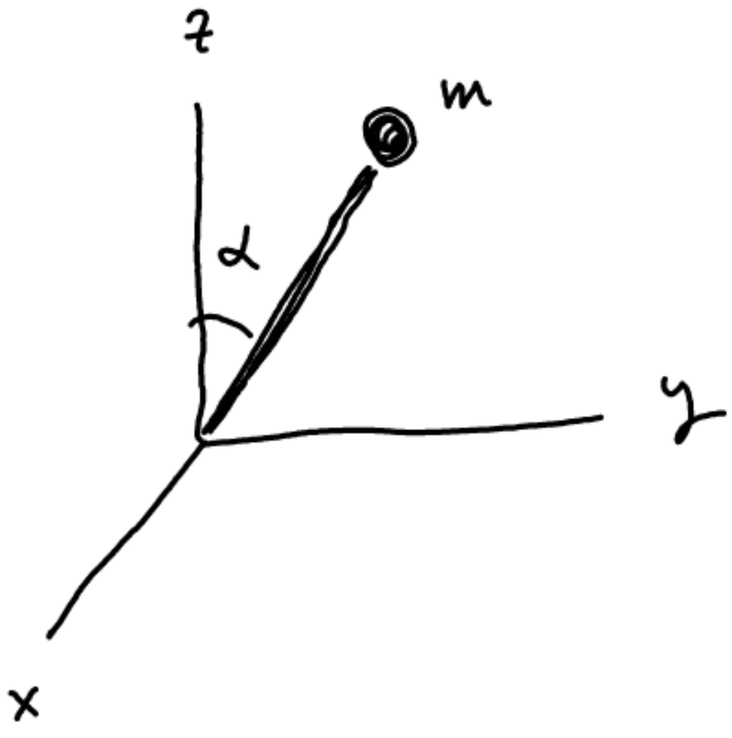
$$\vec{L} = \underline{I}\,\vec{w}$$

I can be written in <u>any</u> basis

<u>BUT</u> there is a special basis where I is <u>diagonal</u> → I_{ij} for $i \neq j = 0$

$\vec{L}$ is in the same direction as $\vec{\omega}$

PRINCIPLE AXES



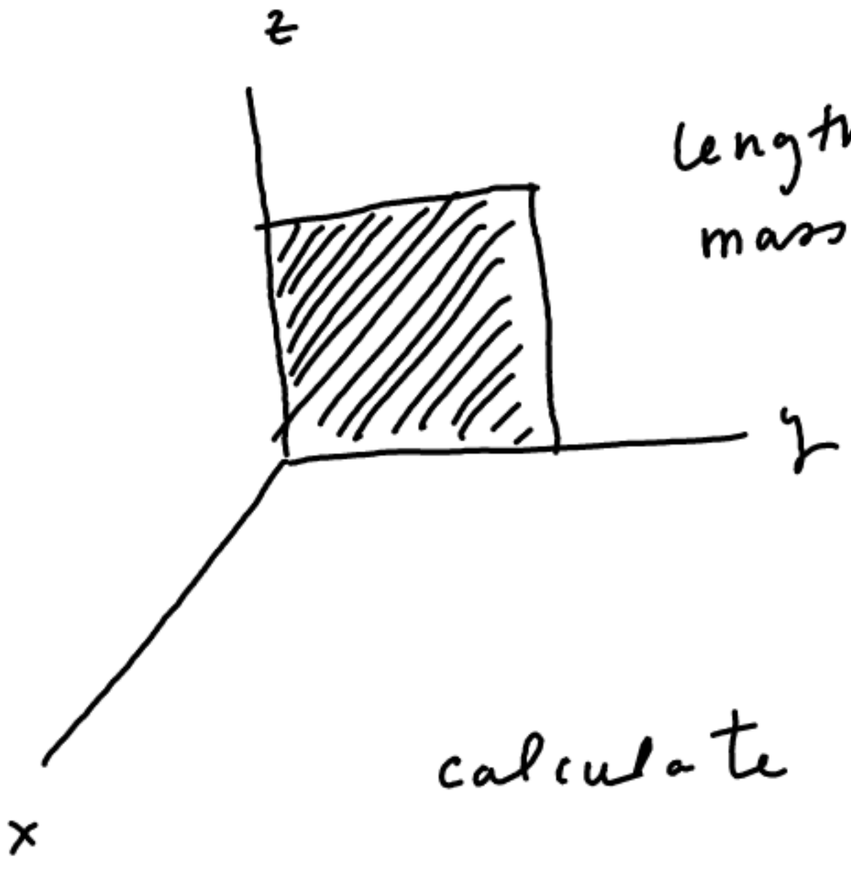
What are the principle axes?

1) along massless rod

2+3) 2 planes containing massless rod

There are always 3 unique principle axes

Rotating a square



length a
mass M

$\rightarrow \quad m_\alpha = \rho = \frac{M}{a^2}$

calculate $\underline{\underline{I}}$

$$I_{xx} = \sum_\alpha m_\alpha (y^2 + z^2)$$

$$= \int_0^a dy \int_0^a dz \, \rho (y^2 + z^2)$$

$$= \rho \int_0^a \left(y^2 a + \frac{1}{3} a^3 \right) dy$$

$$= \rho \left(\frac{1}{3} a^4 + \frac{1}{3} a^4 \right) = \frac{M}{a^2} \frac{2}{3} a^4$$

$$I_{xx} = \frac{2}{3} M a^2$$

$$I_{yy} = \int_0^a dy \int_0^a dz \; \rho (\cancel{x^2}^{\,0} + z^2)$$

$$= \rho a \int_0^a z^2 \, dz$$

$$= \frac{M}{a^2} \cdot a \frac{1}{3} a^3 = \frac{1}{3} M a^2$$

$$I_{yy} = I_{zz} = \frac{1}{3} M a^2$$

$$I_{xy} = I_{yx} = -\int_0^a dy \int_0^a dz \;\; \rho x y \;\; = 0$$

$$I_{xz} = I_{zx} = 0$$

$$I_{yz} = I_{zy} = -\int_0^a dy \int_0^a dz \;\; \rho y z$$

$$= -\int_0^a dy \;\; \frac{1}{2} \rho y a^2$$

$$= -\frac{1}{4} \rho a^4 = -\frac{1}{4} M a^2$$

$$\underline{\underline{I}} = \begin{pmatrix} \frac{2}{3} M a^2 & 0 & 0 \\ 0 & \frac{1}{3} M a^2 & -\frac{1}{4} M a^2 \\ 0 & -\frac{1}{4} M a^2 & \frac{1}{3} M a^2 \end{pmatrix}$$

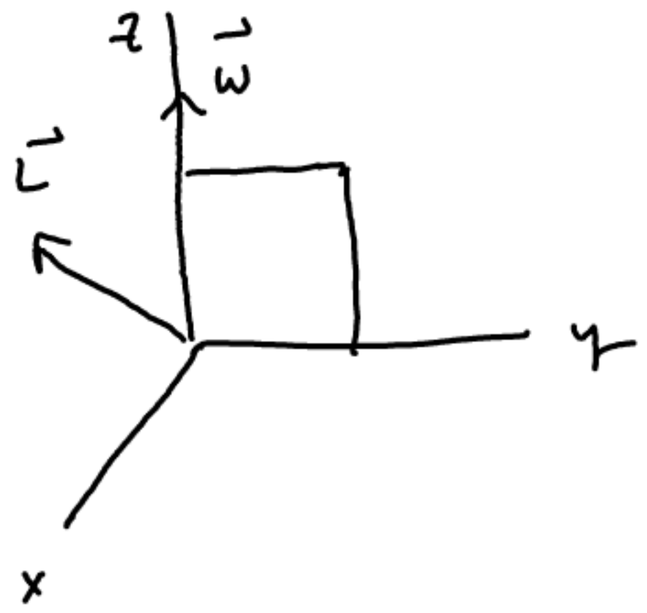
Rotate around z-axis:

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega}$$

$$= Ma^2 \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & -1/4 \\ 0 & -1/4 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

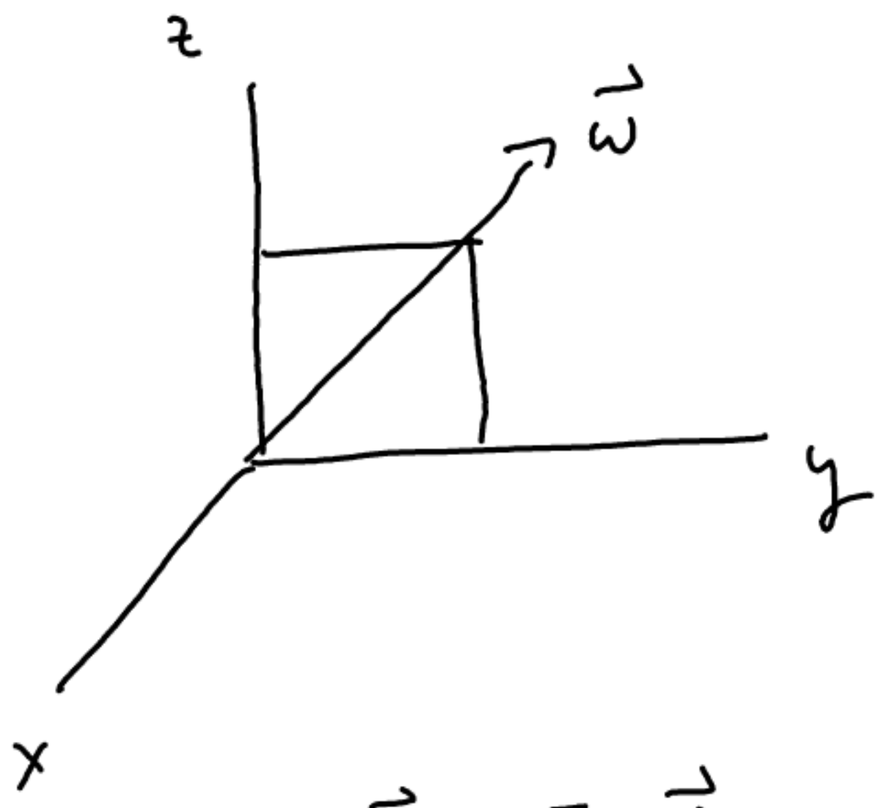
$$\vec{L} = Ma^2 \begin{pmatrix} 0 \\ -\omega/4 \\ \omega/3 \end{pmatrix}$$

some y and z component



not a principle axis

Rotate on diagonal



$$\vec{\omega} = (0, \cos \pi/4, \sin \pi/4) \cdot \omega$$

$$= (0, \omega/\sqrt{2}, \omega/\sqrt{2})$$

$$\vec{L} = \underline{\underline{I}}\, \vec{\omega}$$

$$= \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & +1/3 & -1/4 \\ 0 & -1/4 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{\omega}{3\sqrt{2}} - \frac{\omega}{4\sqrt{2}} \\ \frac{-\omega}{4\sqrt{2}} + \frac{\omega}{3\sqrt{2}} \end{pmatrix}$$

y and z are equal

↓

same direction as $\vec{\omega}$

The diagonal is a principle axis

Finding the Principle Axes of Inertia

We want to diagonalize $\underline{\underline{I}}$ so that

$$\underline{\underline{I}} = \begin{pmatrix} \lambda_1 \omega & 0 & 0 \\ 0 & \lambda_2 \omega & 0 \\ 0 & 0 & \lambda_3 \omega \end{pmatrix}$$ in the correct basis.

or $$\underline{\underline{I}}\,\vec{\omega} = \lambda \vec{\omega}$$

starting basis x, y, z

$$(\underline{\underline{I}} - 1\lambda)\,\vec{\omega} = 0$$

$$\downarrow$$

$$\det(\underline{\underline{I}} - 1\lambda) = 0$$

$$\det \begin{pmatrix} I_{xx} - \lambda_1 & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda_2 & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda_3 \end{pmatrix} = 0$$

Try it for the square:

$$\det\left(\underline{\underline{I}} - 1\lambda \right) = 0$$

$$\det\left[\begin{pmatrix} 2/3 - \lambda & 0 & 0 \\ 0 & 1/3 - \lambda & -1/4 \\ 0 & -1/4 & 1/3 - \lambda \end{pmatrix} \cdot Ma^2 \right] = 0$$

$$\left(2/3 - \lambda \right)\left(\left(1/3 - \lambda \right)\left(1/3 - \lambda \right) - 1/16 \right) = 0$$

$$\left(2/3 - \lambda \right)\left(\left(1/3 - \lambda \right)^2 - 1/16 \right) = 0$$

$\lambda_1 = \frac{2}{3}$ first solution

$\left(1/3 - \lambda \right) = \pm \frac{1}{4}$

$\lambda_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ second soln

$\lambda_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ third soln

To find the principle axes $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$:

$$(\underline{I} - 1\lambda)\,\vec{\omega} = 0$$

(1)
$$\begin{pmatrix} 2/3 & -2/3 & 0 \\ 0 & 1/3 & -2/3 & -1/4 \\ 0 & -1/4 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = 0$$

$$-\frac{1}{3}\omega_{1y} - \frac{1}{4}\omega_{1z} = 0$$

$$-\frac{1}{4}\omega_{1y} - \frac{1}{3}\omega_{1z} = 0$$

$\omega_{1y} = \omega_{1z} = 0$ ω_{1x} is free $\rightarrow = 1$

$\vec{\omega}_1 = (1, 0, 0)\, Ma^2$

(2) $$\begin{pmatrix} 2/3 - 7/12 & 0 & 0 \\ 0 & 1/3 - 7/12 & -1/4 \\ 0 & -1/4 & 1/3 - 7/12 \end{pmatrix} \begin{pmatrix} w_{2x} \\ w_{2y} \\ w_{2z} \end{pmatrix} = 0$$

$$\left(\frac{2}{3} - \frac{7}{12} \right) w_{2x} = 0 \longrightarrow w_{2x} = 0$$

$$1/3 - 7/12 = -1/4 \curvearrowright \quad -\frac{1}{4} w_{2y} - \frac{1}{4} w_{2z} = 0$$

$$-\frac{1}{4} w_{2y} - \frac{1}{4} w_{2z} = 0$$

$$w_{2y} = -w_{2z}$$

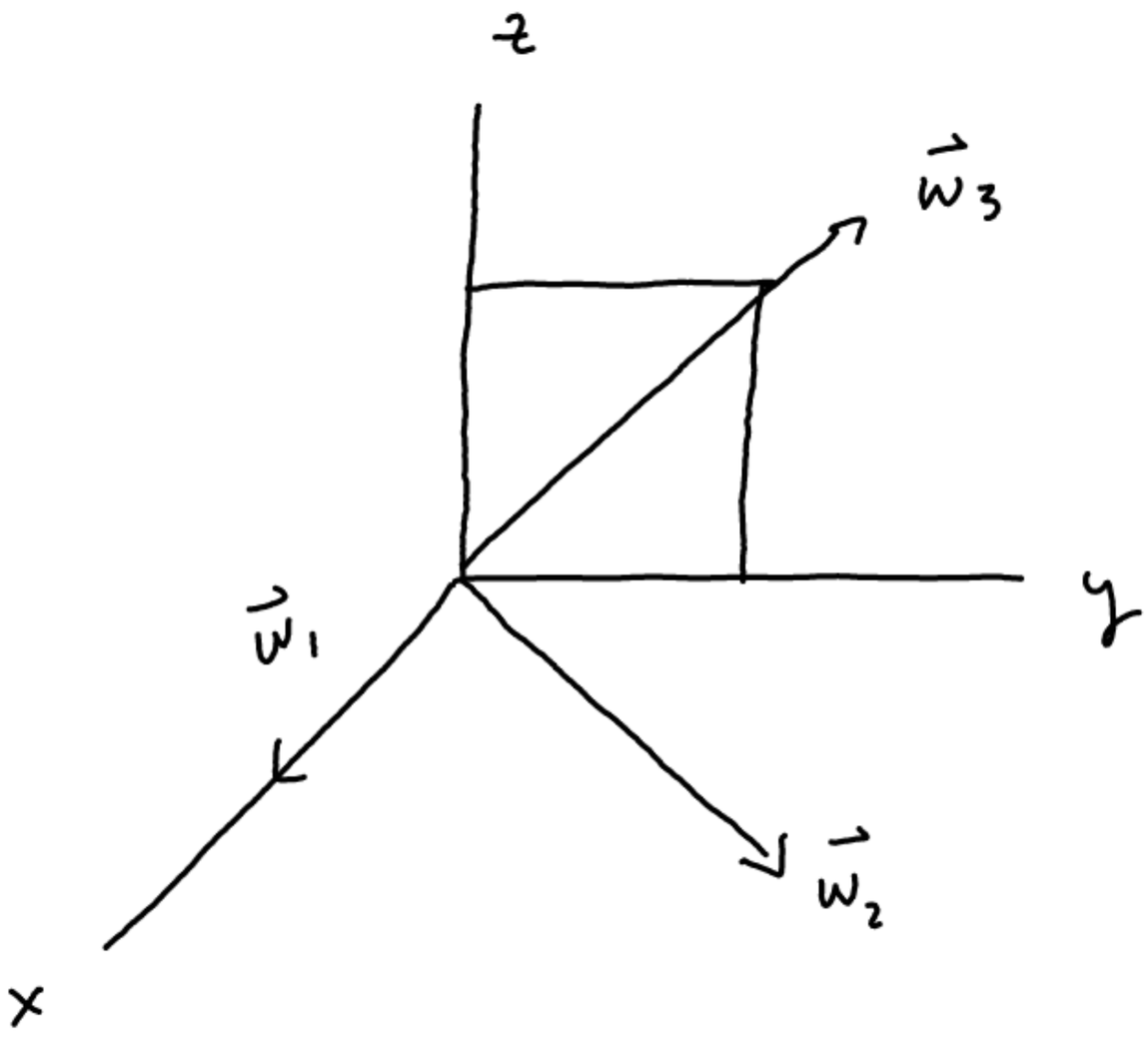
$$\vec{w}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) Ma^2$$

(3)
$$\begin{pmatrix} 2/3 - 1/12 & 0 & 0 \\ 0 & 1/3 - 1/12 & -1/4 \\ 0 & -1/4 & 1/3 - 1/12 \end{pmatrix} \begin{pmatrix} w_{3x} \\ w_{3y} \\ w_{3z} \end{pmatrix} = 0$$

$$w_{3x} = 0$$

$$\frac{1}{3} - \frac{1}{12} = \frac{1}{4} \longrightarrow \quad w_{3y} = w_{3z}$$

$$\vec{w}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) Ma^2$$



all three principle axes