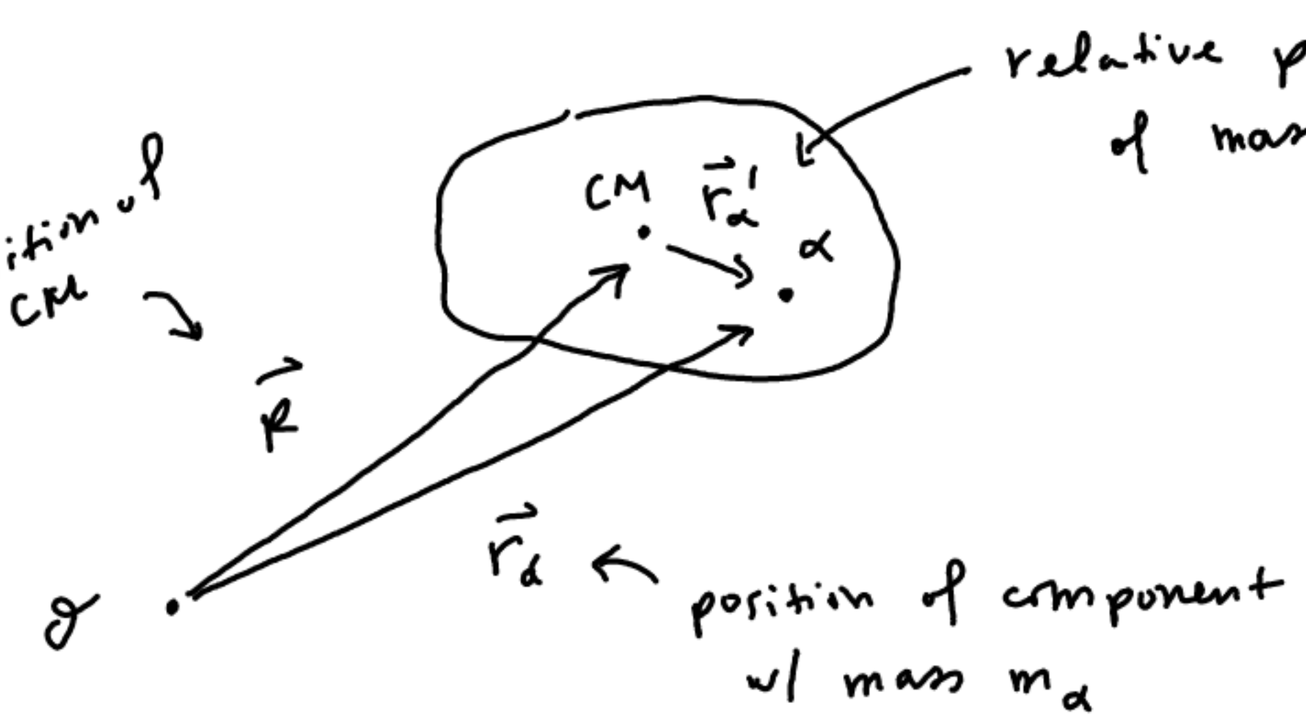
Rotational Coordinates

Just like in our central force problems, we will separate the motion in a rigid body into a CM and relative (rotational) component.

A rigid body is made up of N particles ($\alpha = 1, \ldots, N$)

Each particle has mass m_α

position $\vec{r}_\alpha$



$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$$

definition of CM: $\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha = \frac{1}{M}\int \vec{r}\, dm$

IMPORTANT: for a RIGID BODY $\dot{\vec{r}}_\alpha = 0$

the particles have a fixed relationship to CM.

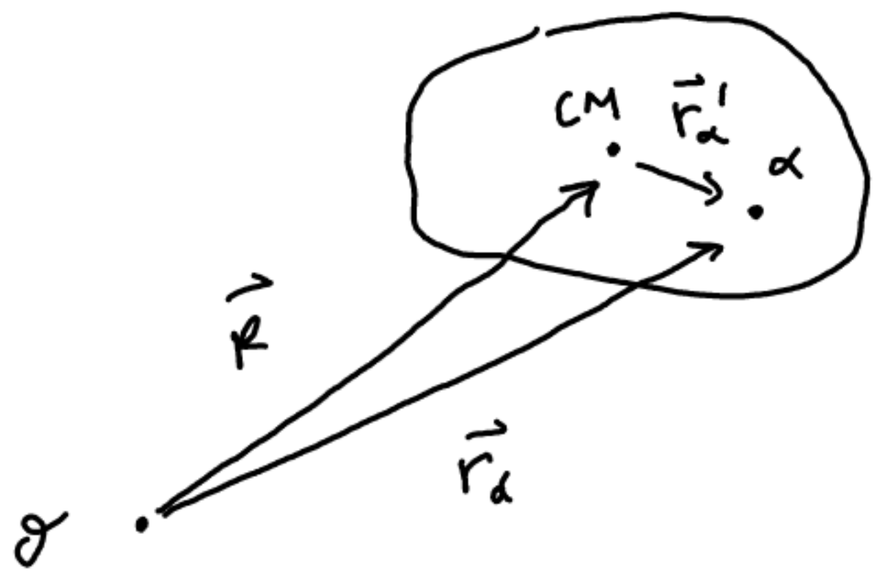
Total Momentum + linear motion

We already know that CM will move like a single particle:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$$ total momentum (CM)

$$\dot{\vec{P}} = \vec{F}_{ext} = M\ddot{\vec{R}}$$

Total Angular Momentum



$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$$

for each particle, $\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times m\dot{\vec{r}}_\alpha$

In total, $\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times m\dot{\vec{r}}_\alpha$

$$\vec{L} = \sum_\alpha (\vec{R} + \vec{r}_\alpha')\times m_\alpha(\dot{\vec{R}} + \dot{\vec{r}}_\alpha')$$

$$\vec{L} = \sum \vec{R}\times m_\alpha\dot{\vec{R}} + \sum \vec{R}\times m_\alpha\dot{\vec{r}}_\alpha' + \sum \vec{r}_\alpha'\times m\dot{\vec{R}} + \sum \vec{r}_\alpha'\times m\dot{\vec{r}}_\alpha'$$

$$= \vec{R}\times M\dot{\vec{R}} + \vec{R}\times \sum m_\alpha\dot{\vec{r}}_\alpha' + \underbrace{\left(\sum \vec{r}_\alpha' m_\alpha\right)}_{\text{CM rel CM} = 0}\times\dot{\vec{R}} + \sum \vec{r}_\alpha'\times m\dot{\vec{r}}_\alpha'$$

$\frac{d}{dt}0 = 0$

$$\vec{L} = \vec{R} \times \vec{P} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$$

$\vec{L}_{CM}$ ↑ $\vec{R} \times \vec{P}$ — (L_{orb})

$\vec{L}_{rel\ CM}$ ↑ $\sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha'$ — (L_{spin})

Kinetic Energy can also be divided

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$$

$$T = \sum_\alpha \frac{1}{2} m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha' \right)^2$$

$$= \sum \frac{1}{2} m\alpha \dot{\vec{R}}^2 + \sum \frac{1}{2} m\alpha \dot{\vec{r}}_\alpha'^{\,2} + \sum m_\alpha \dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha'$$

"

$$\dot{\vec{R}} \underbrace{\sum m_\alpha \dot{\vec{r}}_\alpha'}_{\text{CM rel CM}} = 0$$

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum m_\alpha \dot{\vec{r}}_\alpha'^{\,2}$$

$\frac{1}{2} M \dot{\vec{R}}^2$ ↑ CM motion

$\frac{1}{2} \sum m_\alpha \dot{\vec{r}}_\alpha'^{\,2}$ ↑ motion rel CM

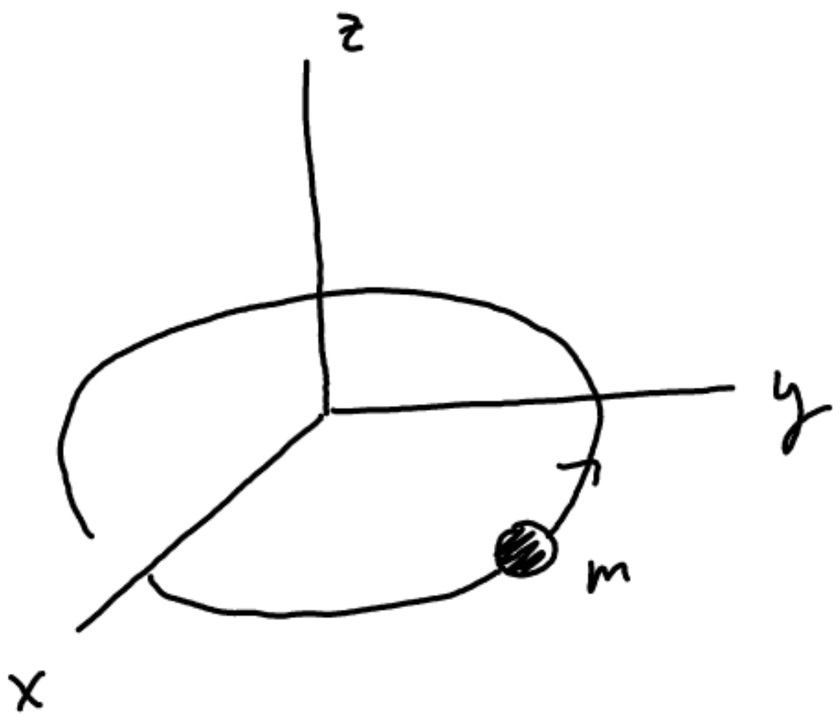
Potential Energy

$$U = U^{ext} + U^{int}$$

U^{ext} ↑ depends on CM

U^{int} ↑ fixed constant

Fixed Axis Rotation



single particle mass m rotating about z-axis

$$\vec{\omega} = \omega \hat{z} = (0, 0, \omega)$$

$$\vec{r} = (x, y, 0)$$

$$\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times m(\vec{\omega} \times \vec{r})$$

$$= \vec{r} \times m(-y\omega, x\omega, 0)$$

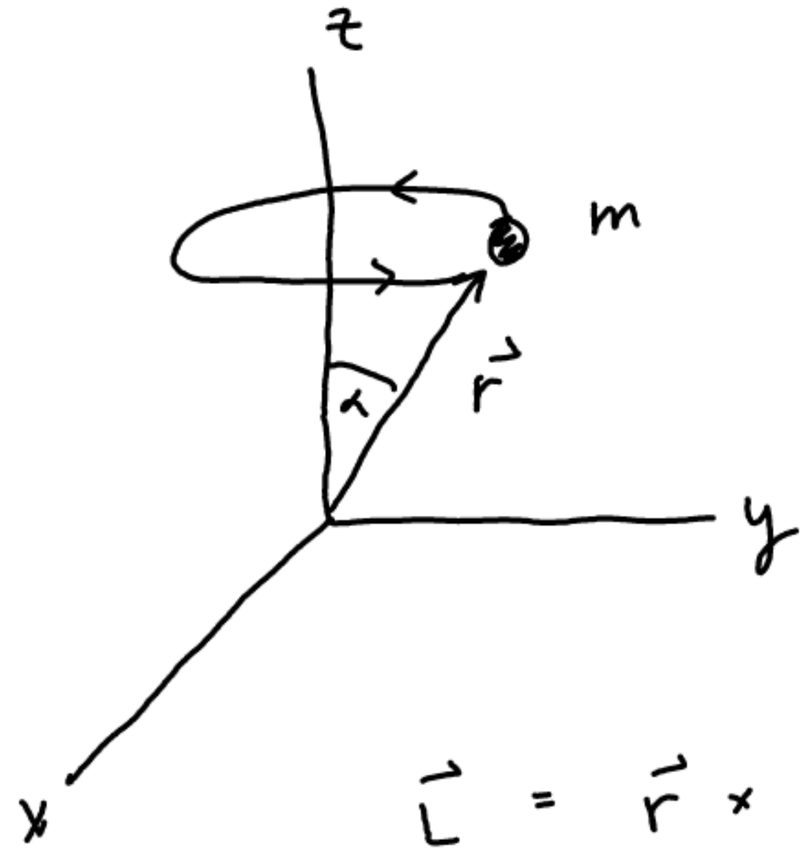
$$\begin{vmatrix} i & j & k \\ x & y & 0 \\ -y\omega & x\omega & 0 \end{vmatrix}$$

$$= m(x^2\omega + y^2\omega)\hat{z}$$

$$\vec{L} = m\rho^2 \omega \hat{z}$$

$$\vec{L} = I\vec{\omega} \qquad I = m\rho^2 \text{ moment of inertia}$$

Now suppose the mass is above the x-y plane, suspended by some massless rod at angle α to z-axis



$$\vec{\omega} = \omega \hat{z} = (0, 0, \omega)$$

$$\vec{r} = (x, y, z)$$

$$\begin{vmatrix} i & j & k \\ x & y & z \\ -y\omega & x\omega & 0 \end{vmatrix}$$

$$\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times m(\vec{\omega} \times \vec{r})$$

$$= \vec{r} \times m(-y\omega, x\omega, 0)$$

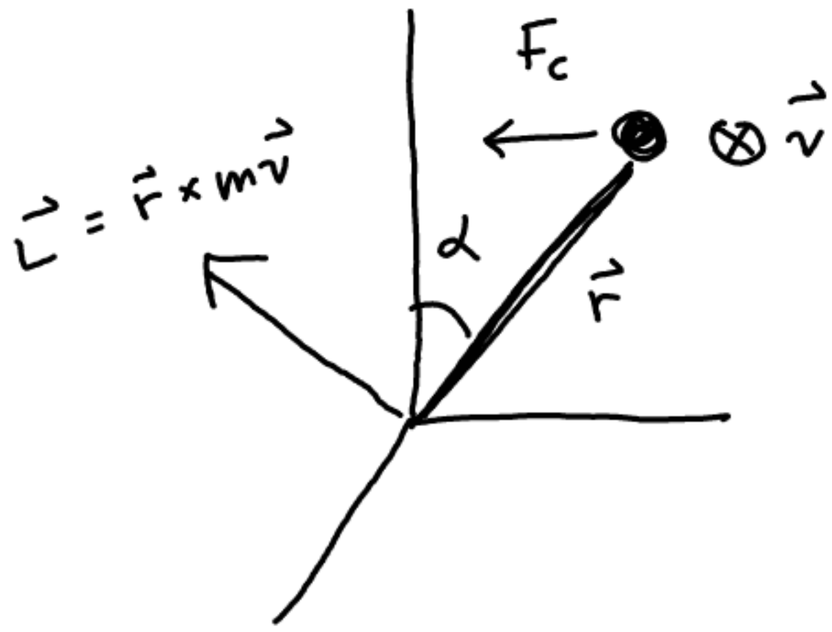
$$= m\omega(-xz, -yz, x^2 + y^2)$$

$L_z = m(x^2+y^2)\omega = m\rho^2\omega$ same as before

$L_x = -mxz\omega$ } non-zero!

$L_y = -myz\omega$ }

Why are there nonzero L_x and L_y?



to go in a circle there must be some F_c

$\downarrow$

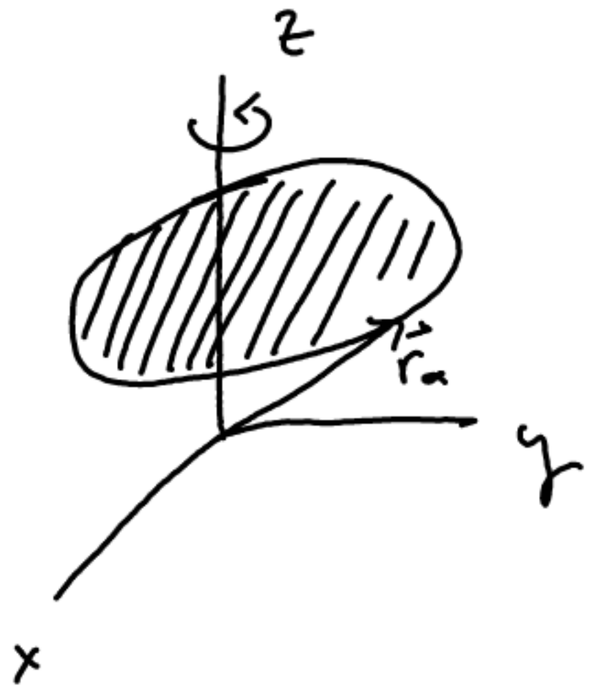
$\vec{\tau} = \vec{r} \times \vec{F}_c$

points in $-\hat{x}$ $\otimes$

Then $\vec{\tau} = \frac{d\vec{L}}{dt}$ so $\vec{L}$ must have $\hat{x}$ component and <u>rotate</u>

The z-offset matters when we have <u>many</u> particles at different z_α (a rotating rigid body)

Now suppose we have a rigid body which will have parts <u>above</u> and/or below the z-axis.



rotate about z-axis

$$\vec{\omega} = \omega \hat{z} = (0, 0, \omega)$$

$$\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$\begin{vmatrix} i & j & k \\ x & y & z \\ -y & x & 0 \end{vmatrix}$$

$$= \sum_\alpha \vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha)$$

$$= \sum_\alpha \vec{r}_\alpha \times m_\alpha (-y_\alpha \omega, +x_\alpha \omega, 0)$$

$$\vec{L} = \sum_\alpha m_\alpha \omega (-x_\alpha z_\alpha, -y_\alpha z_\alpha, x_\alpha^2 + y_\alpha^2)$$

$$L_z = \sum_\alpha m_\alpha \omega (x_\alpha^2 + y_\alpha^2)$$

$$= \sum_\alpha m_\alpha \rho_\alpha^2 \omega$$

z component

same form

$$L_x = -\sum_\alpha m_\alpha x_\alpha z_\alpha \omega$$

$$L_y = -\sum_\alpha m_\alpha y_\alpha z_\alpha \omega$$

new components

$L_x, L_y \neq 0$

Define Products of Inertia:

$$L_x = -\sum_\alpha m_\alpha x_\alpha z_\alpha \omega = I_{xz}\,\omega$$

gives x-component of L when $\vec{\omega}$ is in $\hat{z}$ $\longrightarrow$ $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

$$L_y = -\sum_\alpha m_\alpha y_\alpha z_\alpha \omega = I_{yz}\,\omega$$

$$L_z = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega = I_{zz}\,\omega$$

$$I_{zz} = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)$$

$$\vec{L} = (I_{xz}\omega,\ I_{yz}\omega,\ I_{zz}\omega)$$

for a body rotating about the z-axis