# The Foucault Pendulum

A very long pendulum hangs on the rotating earth.

In an inertial reference frame,

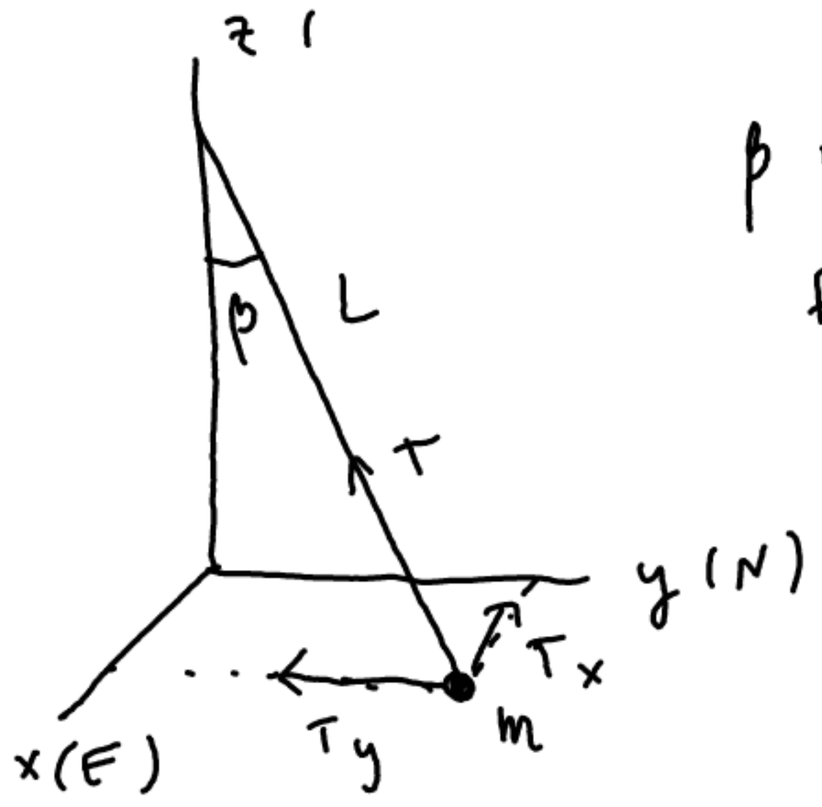
$$m\ddot{\vec{r}}_0 = \vec{T} + m\vec{g}_0$$

In our rotating reference frame,

$$m\ddot{\vec{r}} = \vec{T} + \underbrace{m\vec{g}_0 + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}} + 2m\dot{\vec{r}}\times\vec{\Omega}$$

let the combined effect of $F_g$ and $F_{CF}$ be $m\vec{g}$

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} + 2m\dot{\vec{r}}\times\vec{\Omega}$$




$\beta$ is small $\rightarrow$ $T_z = T\cos\beta \sim T$

for small osc, $T \sim mg$

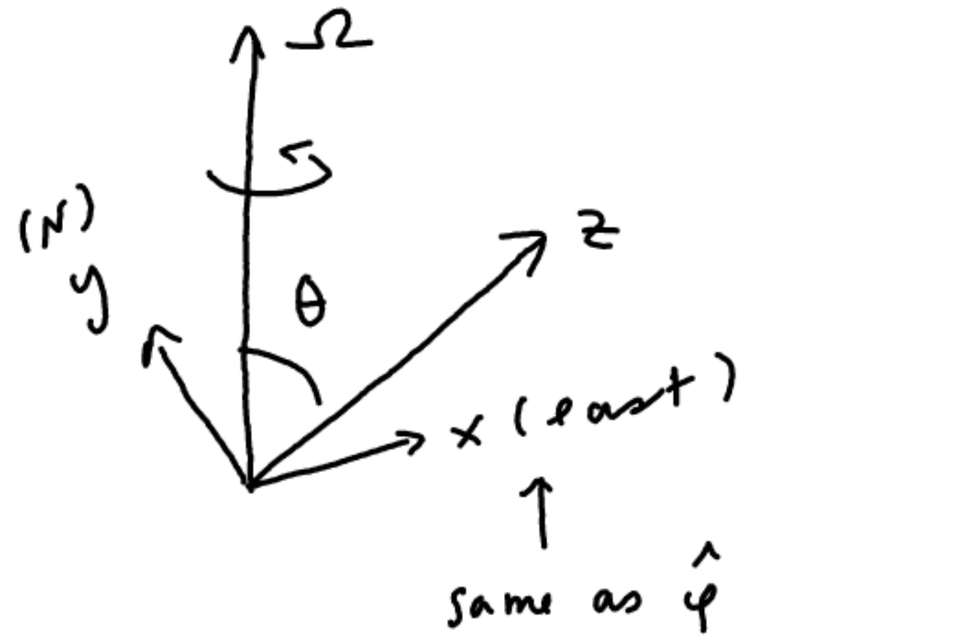
$$\frac{T_x}{T} = -\frac{x}{L} \qquad \frac{T_y}{T} = -\frac{y}{L}$$

$$T_x = -mgx/L \qquad T_y = -mgy/L$$

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}$$

$\vec{T}$: $x, y, z$ — $m\vec{g}$: $z$

$\vec{\Omega} = (0, \sin\theta, \cos\theta)\cdot\Omega$




no $\Omega$ component in $\varphi$

$$\dot{\vec{r}} \times \vec{\Omega} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

$$\hookrightarrow (\dot{y}\cos\theta - \dot{z}\sin\theta)\hat{i}\cdot\Omega + (0 - \dot{x}\cos\theta)\hat{j}\cdot\Omega + (\dot{x}\sin\theta - 0)\hat{k}\cdot\Omega$$

gather terms: ($\dot{z}$ small, $\dot{z}\sin\theta \approx 0$)

$$m\ddot{x} = -mgx/L + 2m\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta)$$

$$m\ddot{y} = -mgy/L - 2m\Omega\dot{x}\cos\theta$$

ignore $z$ eqn

$$\ddot{x} = -g\frac{x}{L} + 2\dot{y}\Omega\cos\theta$$

$$\ddot{y} = -g\frac{y}{L} - 2\dot{x}\Omega\cos\theta$$

let $g/L = \omega_0^2$ pendulum frequency

let $\Omega\cos\theta = \Omega_z$

$$\ddot{x} + \omega_0^2 x - 2\dot{y}\Omega\cos\theta = 0$$

$$\ddot{y} + \omega_0^2 y + 2\dot{x}\Omega\cos\theta = 0$$

inhomogeneous differential equation

introduce change of variables $\eta = x + iy$

$$\ddot{x} + i\ddot{y} + \omega_0^2(x + iy) - 2\Omega_z(\dot{y} - i\dot{x}) = 0$$

$$\ddot{\eta} + \omega_0^2\eta + 2i\,\Omega_z\theta\dot{\eta} = 0$$

guess: $\eta = C_1 e^{-i\alpha t}$

$$\ddot{\eta} = C_1(-\alpha^2)\eta$$

$$\dot{\eta} = C_1(i\alpha)\eta$$

$$-\alpha^2\eta + (2i\,\Omega_z)(i\alpha)\eta + \omega_0^2\eta = 0$$

$$-\alpha^2 - 2\alpha\Omega_z + \omega_0^2 = 0$$

$$\alpha = (2\Omega_z \pm \sqrt{4\Omega_z^2 - 4(-1)\omega_0^2})/2$$

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2}$$

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + w_0^2} \sim \Omega_z \mp w_0$$

bic $\Omega_z << w_0$

thus

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{iw_0 t} + C_2 e^{-iw_0 t}\right)$$

$$= e^{-i\Omega_z t} \cdot A\cos(w_0 t + B)$$

$$\eta = (\cos\Omega_z t - i\sin\Omega_z t)\cdot A\cos(w_0 t + B)$$

$$\eta = x + iy$$

$$\hookrightarrow \quad x = A\cos\Omega_z t\cos(w_0 t + B)$$

$$y = -A\sin\Omega_z t\cos(w_0 t + B)$$

Since $w_0 >> \Omega_z$ : starts osc. at $w_0$ in $x$

switches over to osc at $w_0$ in $y$

