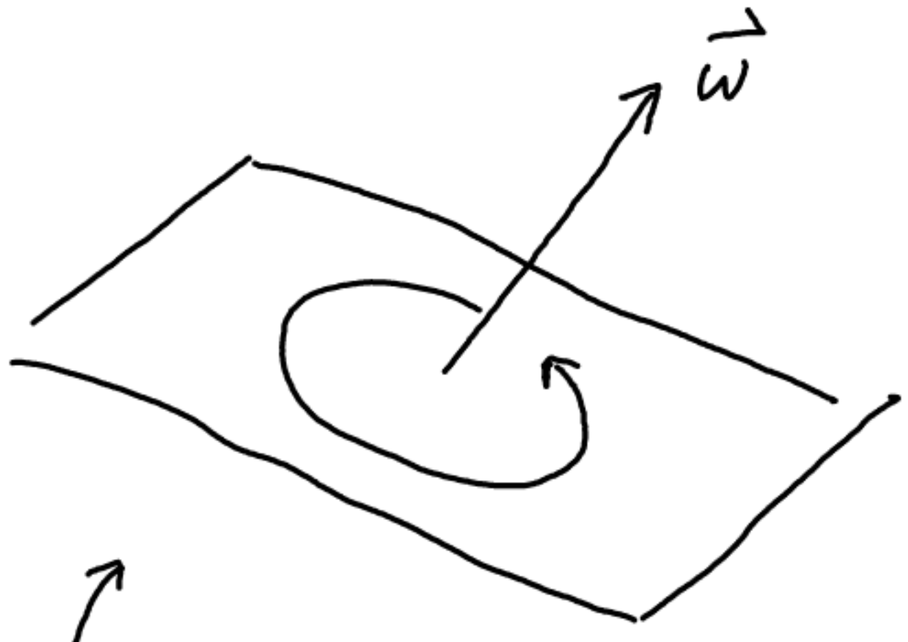
Rotating Frames



right hand rule

angular velocity vector $\vec{\omega}$

- defines plane of rotation
- unambiguous direction of rotation
- angular speed

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

let $\hat{e}$ be a unit vector in a rigid body

then $\frac{d\hat{e}}{dt} = \vec{\omega} \times \vec{e}$ ← more general than $\vec{v}$

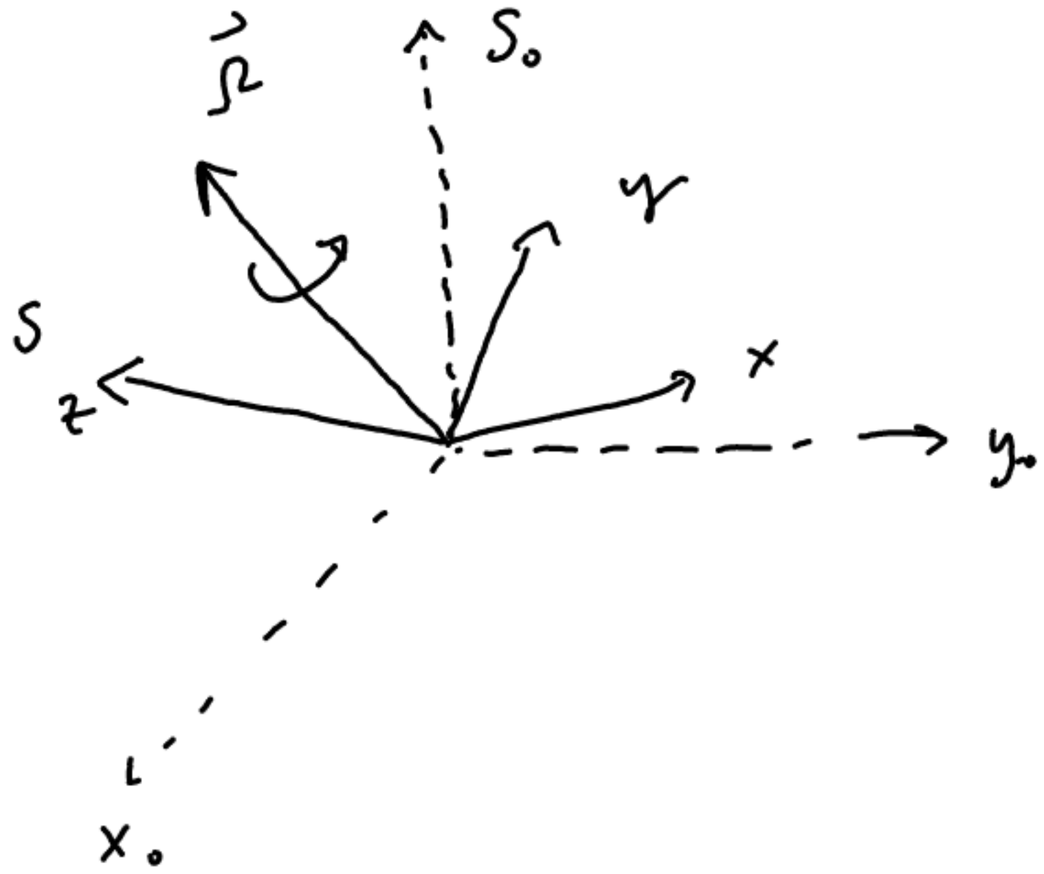
If $\vec{\omega}$ is the angular velocity of a body

then let $\vec{\Omega}$ be the angular velocity of a reference frame.

Time derivatives in rotating frame

S and S_0 share a common origin

S rotates w/ angular velocity $\vec{\Omega}$ ← constant



any vector $\vec{Q}$ in S

may change in time

rel. to either S or S_0.

$\left(\frac{d\vec{Q}}{dt}\right)_S$ $\qquad$ $\left(\frac{d\vec{Q}}{dt}\right)_{S_0}$

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be an orthogonal basis in S:

$$\vec{Q} = Q\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3$$

$$\left(\frac{dQ}{dt}\right)_S = \sum_i \frac{dQ_i}{dt}\hat{e}_i$$

$$\left(\frac{dQ}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q\frac{d\hat{e}_i}{dt}$$ ← $\hat{e}_i$ move in S_0

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q\frac{d\hat{e}_i}{dt}$$

by our earlier definition $\frac{d\hat{e}}{dt} = \vec{\omega}\times\hat{e}$

then $\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \vec{\Omega}\times\hat{e}_i$

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_i \frac{dQ_i}{dt}\hat{e}_i + \sum_i Q_i(\vec{\Omega}\times\hat{e}_i)$$

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_{S} + \vec{\Omega}\times\vec{Q}}$$

we will need this to take derivatives of any coordinate in a rotating frame.

To solve Newton's 2nd law, we need

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$$

Using our rotating time derivative,

$$\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d}{dt}\right)_{S_0}\left(\frac{d\vec{r}}{dt}\right)_{S_0}$$

$$= \left(\frac{d}{dt}\right)_{S_0}\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

apply it again →

$$= \left(\frac{d}{dt}\right)_S\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

$$+ \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_S + \vec{\Omega}\times\vec{r}\right]$$

let $\dot{r}$ be the S time derivative

$$= \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$$

$$m\left(\ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right) = \vec{F}$$

$$m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$

$\vec{F}$ ↑ noninertial forces

$2m\dot{\vec{r}}\times\vec{\Omega}$ ↑ coriolis force $\vec{F}_{cor}$

$m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$ ↑ centrifugal force $\vec{F}_{cf}$