Bound Orbits $\varepsilon < 1$

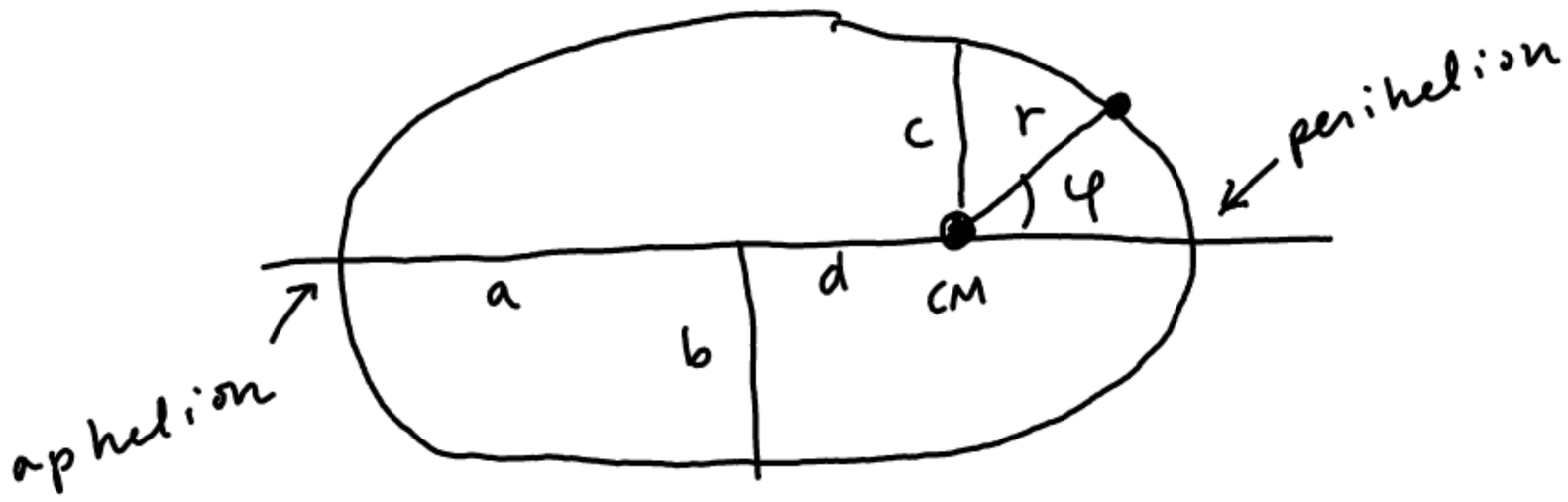
$c = \ell^2/\gamma\mu$

$$r(\varphi) = \frac{c}{1 + \varepsilon\cos\varphi} \longrightarrow \begin{cases} r_{min} = \dfrac{c}{1+\varepsilon} & \varphi = 0 \quad \text{perihelion / perigee} \\ r_{max} = \dfrac{c}{1-\varepsilon} & \varphi = \pi \quad \text{apphelion / apogee} \end{cases}$$

$\varphi = 0$ is always the minimum r-value

cartesian form $\dfrac{(x+d)^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\begin{cases} a = \dfrac{c}{1-\varepsilon^2} & \text{semimajor axis} \\ b = \dfrac{c}{\sqrt{1-\varepsilon^2}} & \text{semiminor axis} \\ d = a\varepsilon & \text{location of CM (focus)} \end{cases}$$



$\vec{r}$ is still the position of our fictitious relative particle.

If we choose something like a planet orbiting the sun, the sun is at CM and the planet is at $\vec{r}$.

definition of ε: eccentricity

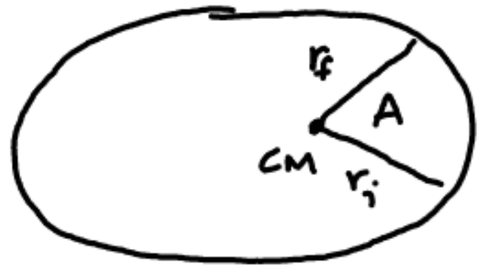
$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

Kepler's First Law

Orbital Period (recovering some time information)

Kepler's 2nd law: $\vec{r}(t)$ sweeps out some area A at a rate

$$\frac{dA}{dt} = \frac{l}{2\mu}$$



total area of ellipse: $A = \pi a b$

period $$\tau = \frac{A}{dA/dt} = 2\pi a b \mu / l$$

recall $$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

$$\tau^2 = 4\pi^2 a^2 \cdot a^2(1-\varepsilon^2)\mu^2/l^2$$

$$= 4\pi^2 a^3 c \mu^2 / l^2 \longleftarrow a = \frac{c}{1-\varepsilon^2}$$

$$\tau^2 = 4\pi^2 a^3 \mu / \gamma \longleftarrow c = l^2/\gamma\mu$$

$\gamma = G m_1 m_2 \sim G\mu m_2$ for $m_2 >> 1$

Kepler's Third Law →

$$\boxed{\tau^2 = \frac{4\pi^2}{G m_2} a^3}$$

where m_2 is the much larger mass.

Unbounded Orbits $\varepsilon \geq 1$

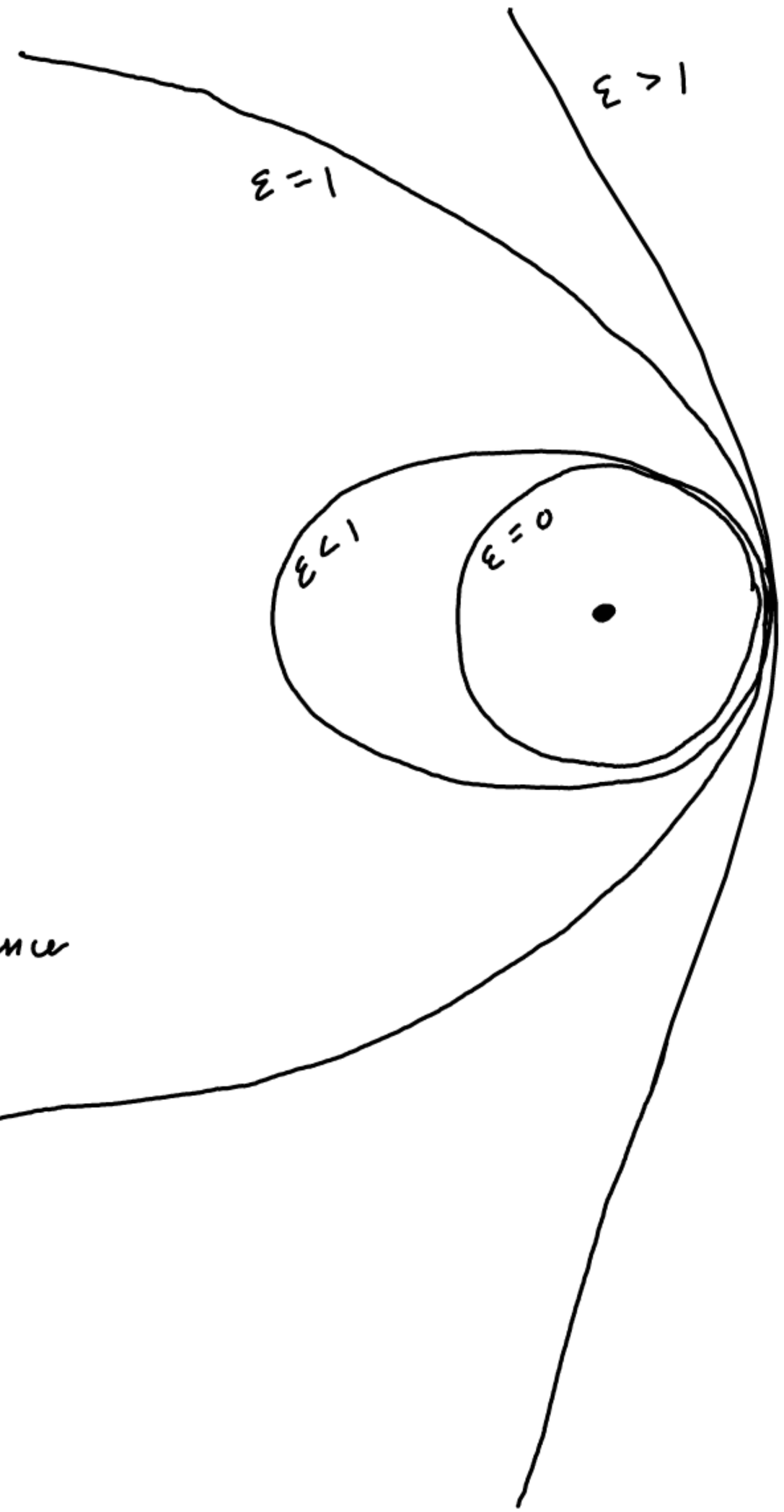
$\varepsilon < 0$: bound ellipses

$\varepsilon = 1$: $r(\varphi) = \dfrac{c}{1+\cos\varphi}$, $r_{min} = \dfrac{c}{2}$

for $\varphi = \pm\pi$, r diverges.

equation for parabola

$$x = \frac{c}{2} - \frac{y^2}{2c}$$



condition for divergence

$\varepsilon > 1$: hyperbola

with $\cos(\varphi_{max}) = -1/\varepsilon$

asymptotes

$$r_{min} = \frac{c}{1+\varepsilon}$$

Orbital Energy + Angular Momentum

Energy increases with ε

- bound $\varepsilon < 1$, $E < 0$
- unbound $\varepsilon > 1$, $E > 0$

$$c = \frac{l^2}{\mu \gamma} \rightarrow l = \sqrt{c \mu \gamma}$$

$$= \sqrt{G m_1 m_2 \mu c}$$

for $m_2 >> m_1$

$m_1 \sim \mu$, $m_2 \sim M$

$$l \sim \mu \sqrt{GMc}$$

kinetic energy vanishes at turning points:
(r_{min}, r_{max})

$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{l^2}{2\mu r_{min}^2}$$

$$r_{min} = \frac{c}{1+\varepsilon} = \frac{l^2}{\mu\gamma(1+\varepsilon)}$$

$$E = -\frac{\gamma^2 \mu (1+\varepsilon)}{l^2} + \frac{l^2 \cdot \mu^2 \gamma^2 (1+\varepsilon)^2}{2\mu l^2 \cdot l^2}$$

$$\boxed{E = +\frac{\gamma^2 \mu}{2l^2}(\varepsilon^2 - 1) = \frac{\gamma}{2c}(\varepsilon^2 - 1)}$$

for $m_1 << m_2$, $\gamma \sim GM\mu$

$$E = \frac{GM\mu}{2c}(\varepsilon^2 - 1)$$

$$= \frac{GM\mu}{2a}$$

orbital speed

$$v = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$$

$$= \sqrt{\left(\frac{dr}{d\varphi}\,\frac{d\varphi}{dt}\right)^2 + r^2\dot{\varphi}^2}$$

$$= \dot{\varphi}\sqrt{\left(\frac{dr}{d\varphi}\right)^2 + r^2}$$

$$r = \frac{c}{1+\varepsilon\cos\varphi}$$

$$\frac{dr}{d\varphi} = \frac{c\varepsilon\sin\varphi}{(1+\varepsilon\cos\varphi)^2}$$

$$v = \dot{\varphi}\sqrt{\frac{c^2\varepsilon^2\sin^2\varphi}{(1+\varepsilon\cos\varphi)^4} + \frac{c^2}{(1+\varepsilon\cos\varphi)^2}}$$

$$= \dot{\varphi}\sqrt{\frac{r^4\varepsilon^2}{c^2}\sin^2\varphi + r^2}$$

$$\dot{\varphi} = l/\mu r^2$$

$$v = \frac{l}{\mu r^2}\frac{r^2}{c}\sqrt{\varepsilon^2\sin^2\varphi + \frac{c^2}{r^2}}$$

$$= \sqrt{\frac{\gamma}{\mu c}}\sqrt{\varepsilon^2\sin^2\varphi + c^2/r^2}$$

$$v \sim \sqrt{\frac{GM}{c}}\sqrt{\varepsilon^2\sin^2\varphi + c^2/r^2}$$

$m_2 >> m_1$

for circular orbits, $\varepsilon = 0$
$c = r$

$$v \sim \sqrt{\frac{GM}{r}}$$

Summary orbital formulas

$$U(r) = -\gamma/r$$

$$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{M}$$

} for $m_1 << m_2$: $m_1 \sim \mu$, $m_2 \sim M$, $\gamma \sim GM\mu$

eqn of orbit $r(\varphi) = \dfrac{c}{1 + \varepsilon \cos\varphi}$

orbital speed $v(\varphi) = \sqrt{\dfrac{\gamma}{\mu c}} \sqrt{\dfrac{c^2}{r^2} + \varepsilon^2 \sin^2\varphi}$

$$\sim \sqrt{\frac{GM}{c}} \sqrt{\frac{c^2}{r^2} + \varepsilon^2 \sin^2\varphi} \quad \text{if } m_1 << m_2$$

orbital energy $E = \dfrac{\gamma}{2c}(\varepsilon^2 - 1)$

angular momentum $l = \sqrt{\gamma \mu c}$

orbital period $\tau^2 = \dfrac{\mu}{\gamma} 4\pi^2 a^3$