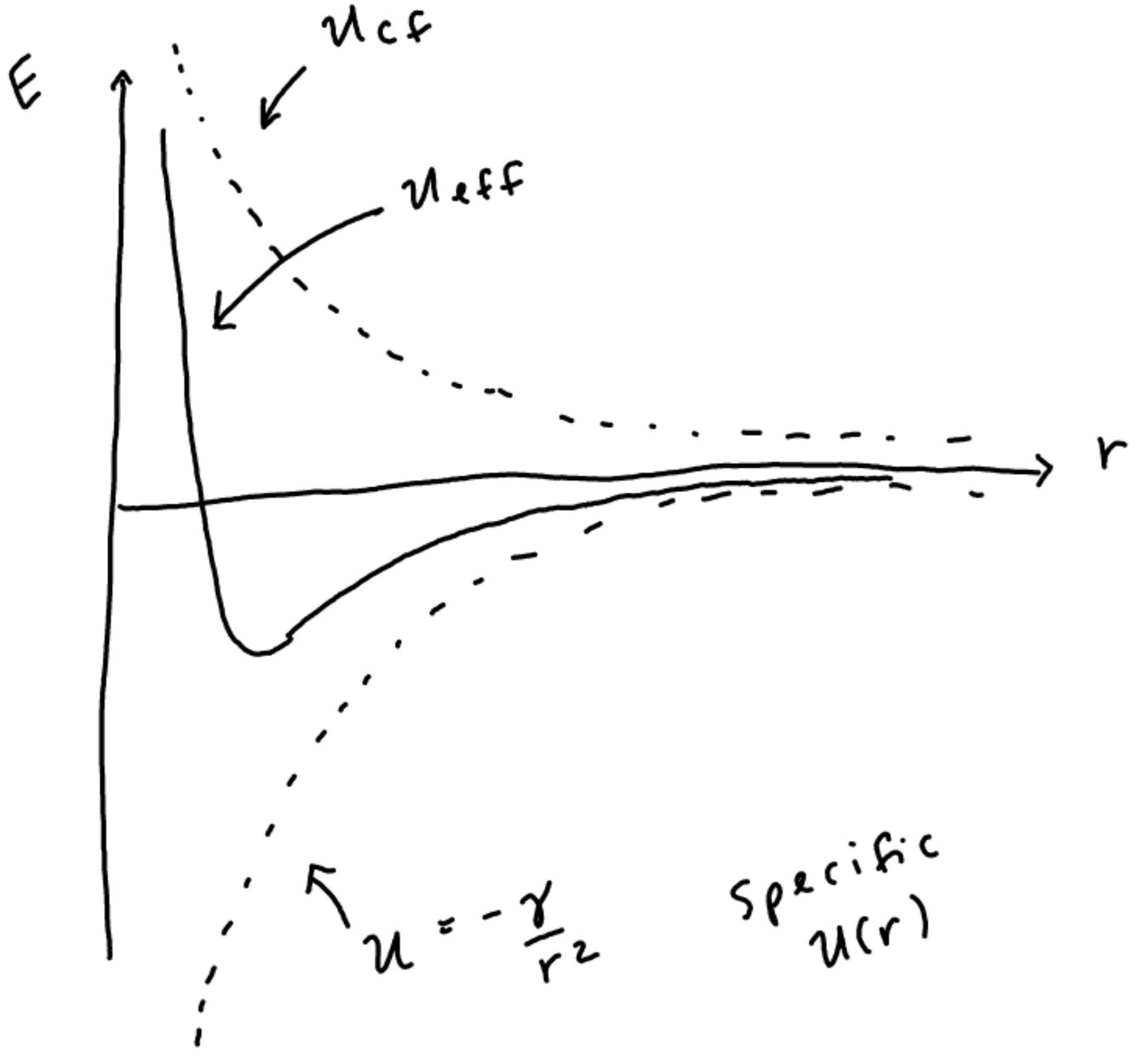
Energy Diagrams + Effective Potential

$$U_{eff} = U(r) + \frac{l^2}{2\mu r^2}$$

- $U(r)$: central force potential
- $\frac{l^2}{2\mu r^2}$: centrifugal potential, U_{cf}

True for any central force

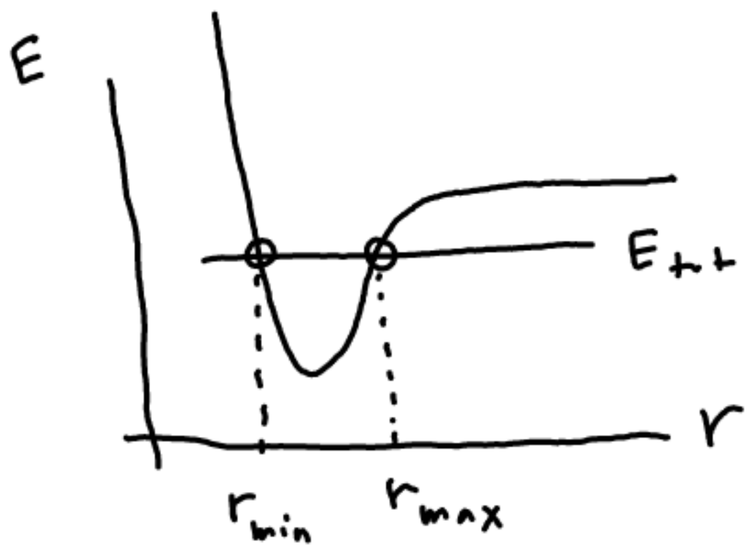
- Gravity
- charged particles
- springs



energy analysis:

- energy is conserved (const.)
- draw in a horizontal line for E
- particle must stay below that line $E = T + U$
- $E < 0$, bound
  - ↳ crossing points $E = U$ get $T = 0$ "turning points"
- $E \geq 0$ unbound
  - ↳ can escape the "well"

Orbits



if $l \neq 0$, we will have a bound orbit

It would be convenient to know $r(\varphi)$

$$\mu \ddot{r} = -\frac{dU}{dr} + \frac{l^2}{\mu r^3} = F(r) + \frac{l^2}{\mu r^3}$$

get out of $\frac{d}{dt} \longrightarrow \frac{d}{d\varphi}$:

simplification trick: $u = \frac{1}{r} \Leftrightarrow r = \frac{1}{u}$

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \dot{\varphi}\frac{d}{d\varphi} = \frac{l}{\mu r^2}\frac{d}{d\varphi} = \frac{l u^2}{\mu}\frac{d}{d\varphi}$$

then $\dot{r} = \frac{d}{dt}(r) = \frac{l u^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right) = \frac{l u^2}{\mu}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right) = -\frac{l}{\mu}\frac{du}{d\varphi}$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{l u^2}{\mu}\frac{d}{d\varphi}\left(-\frac{l}{\mu}\frac{du}{d\varphi}\right) = \frac{-l^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2}$$

Then, $\frac{-l^2u^2}{\mu^2}\frac{d^2u}{d\varphi^2} = F + \frac{l^2u^3}{\mu} \longrightarrow u''(\varphi) = -u(\varphi) - \frac{\mu}{l^2 u(\varphi)^2} F$

Solve $u''(\varphi) = -u(\varphi) - \frac{\mu}{\ell^2 u(\varphi)^2} F$ for $F = -\frac{\gamma}{r^2}$ attractive $\frac{1}{r^2}$ force

$= -\gamma u^2$

$u''(\varphi) = -u(\varphi) - \frac{\mu\gamma}{\ell^2}$ ← removed $\frac{1}{u^2}$ dependence!

$u(\varphi) = \frac{\gamma\mu}{\ell^2} + A\cos\varphi$

$= \frac{\gamma\mu}{\ell^2}\left(1 + \frac{A\ell^2}{\gamma\mu}\cos\varphi\right)$

$u(\varphi) = \frac{\gamma\mu}{\ell^2}(1 + \varepsilon\cos\varphi)$

$r(\varphi) = \frac{1}{u} = \frac{\ell^2}{\gamma\mu}\frac{1}{1+\varepsilon\cos\varphi}$

$$\boxed{r(\varphi) = \frac{c}{1+\varepsilon\cos\varphi}}$$

conic section

$\varepsilon > 0$ always

bound: $\varepsilon < 1$
- ellipse

unbound: $\varepsilon \geq 1$
- parabola $\varepsilon = 1$
- hyperbola $\varepsilon > 1$

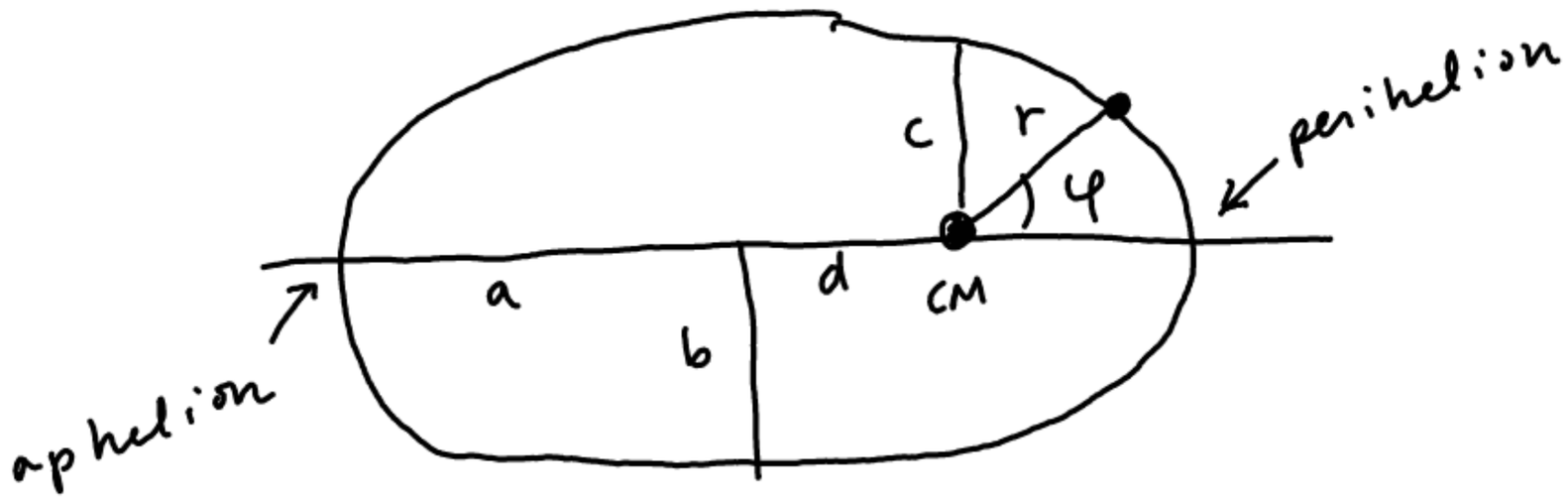
Bound Orbits $\varepsilon < 1$

$\varphi = 0$ is always the minimum r-value

$$r(\varphi) = \frac{c}{1 + \varepsilon \cos\varphi} \longrightarrow \begin{cases} r_{min} = \dfrac{c}{1+\varepsilon} & \varphi = 0 \quad \text{perihelion / perigee} \\ r_{max} = \dfrac{c}{1-\varepsilon} & \varphi = \pi \quad \text{apphelion / apogee} \end{cases}$$

cartesian form $\dfrac{(x+d)^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\begin{cases} a = \dfrac{c}{1-\varepsilon^2} & \text{semimajor axis} \\ b = \dfrac{c}{\sqrt{1-\varepsilon^2}} & \text{semiminor axis} \\ d = a\varepsilon & \text{location of CM (focus)} \end{cases}$$



$\vec{r}$ is still the position of our fictitious relative particle.

If we choose something like a planet orbiting the sun, the sun is at CM and the planet is at $\vec{r}$.

definition of ε : eccentricity

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$

Kepler's First Law