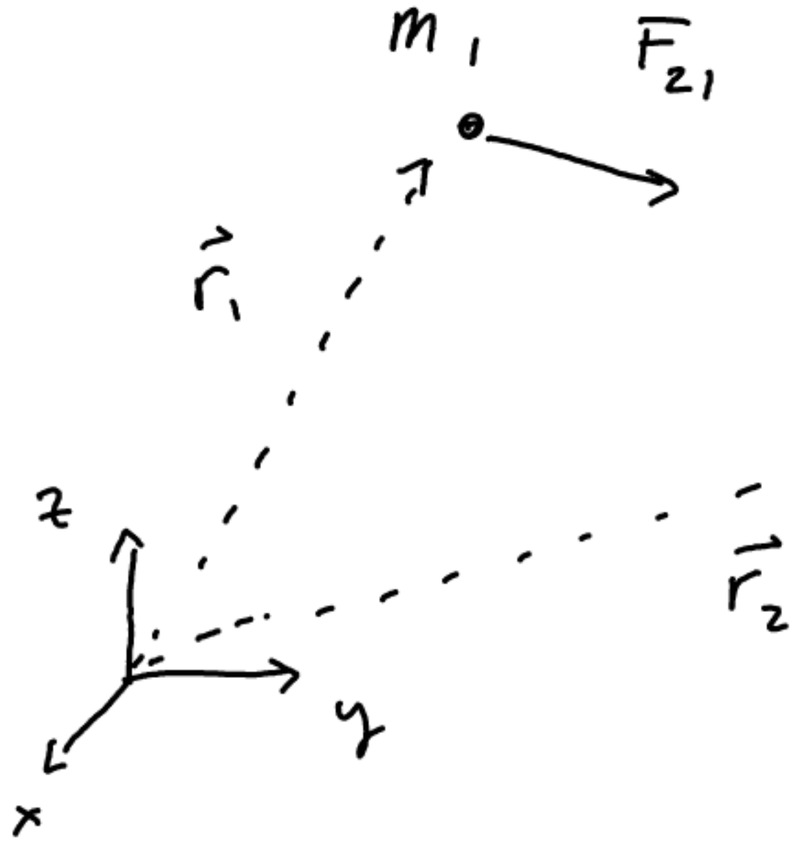
The Two-Body Central Force Problem



If F_{21} and F_{12} are conservative, then they are associated with some potential $U(\vec{r}_1, \vec{r}_2)$.

In a central force problem, U depends on the difference $|\vec{r}_1 - \vec{r}_2|$ (like gravity)

We will instead use <u>relative position</u>

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \longrightarrow U(r)$$

This leaves the Lagrangian

$$\mathcal{L} = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(r)$$

r_1 and r_2 are acceptable but not the most useful coordinates.

The <u>center-of-mass</u> frame has more to offer.

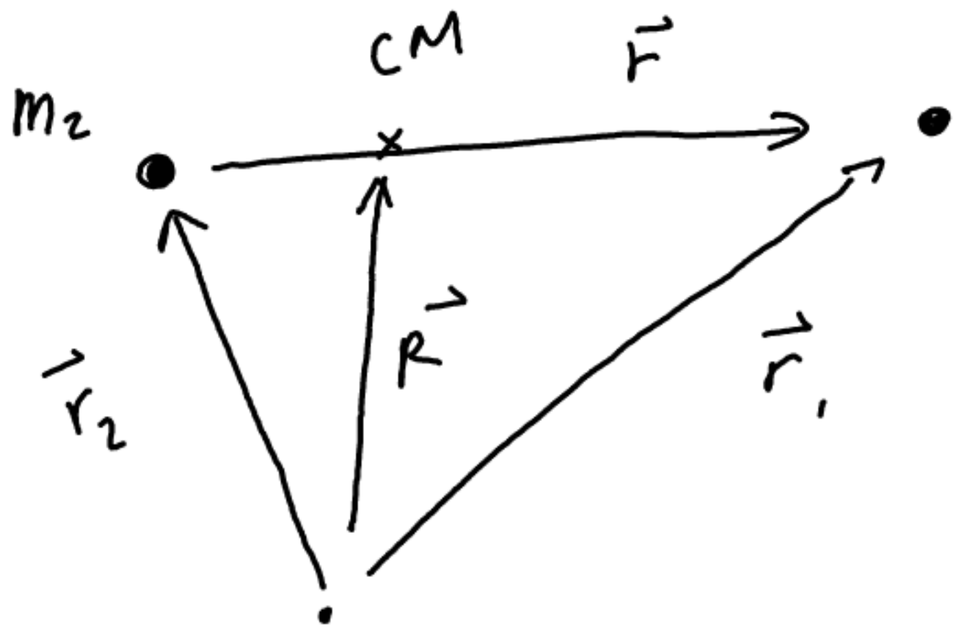
CM and Relative coordinates

If we want one of our generalized coordinates to be $\vec{r}$ (between the two particles), which is a relative position, then we also must have a coordinate to track their absolute position compared to the origin of our coordinate system.

The center-of-mass is a clear option:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

$M \leftarrow$ total mass



note that the ratio of the distances to the CM are m_1/m_2

or

The total momentum of the system is the same as the center of mass momentum:

$$\vec{p}_{tot} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

$$= M \quad \frac{(m_1 \vec{r}_1 + m_2 \vec{r}_2)}{M} = M \vec{R} = \vec{p}_{CM}$$

If the total momentum is conserved, then $\ddot{\vec{R}} = 0$

and $\dot{\vec{R}}$ is constant $\longrightarrow$ inertial reference frame

can choose CM at rest.

(the CM frame)

generalized coordinates:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{r}_2 = \vec{r}_1 - \vec{r}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r})}{M} \qquad \vec{r}_1 = \vec{r} + \vec{r}_2$$

$$\vec{R} = m_1 \frac{(\vec{r} + \vec{r}_2) + m_2 \vec{r}_2}{M}$$

$$\vec{R} = \vec{r}_1 - \frac{m_2}{M} \vec{r} \qquad \vec{R} = \vec{r}_2 + \frac{m_1}{M} \vec{r}$$

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}$$

We can now write our Lagrangian in our general coordinates $\vec{R}$, $\vec{r}$:

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$= \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}} \right)^2$$

$$= \frac{1}{2} (m_1 + m_2) \dot{\vec{R}}^2 + \frac{m_2 m_1}{M} \dot{\vec{r}}^2$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m_1 \frac{m_2^2}{M^2} \dot{\vec{r}}^2 + \frac{1}{2} m_2 \frac{m_1^2}{M^2} \dot{\vec{r}}^2$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} (m_1 + m_2) \frac{m_1 m_2}{M^2} \dot{\vec{r}}^2$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec{r}}^2$$

let $\frac{m_1 m_2}{M} = \mu$ the reduced mass

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2$$

like two fictitious particles

representing CM and rel. position

$$\mathcal{L} = T - U = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(\vec{r})$$

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \left(\frac{1}{2} \mu \dot{\vec{r}}^2 - U(\vec{r}) \right)$$

$$= \mathcal{L}_{CM} + \mathcal{L}_{rel}$$

two separable Lagrangians

$\vec{R}$, $\vec{r}$ can be solved completely separately

$\vec{R}$ equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = M \ddot{\vec{R}} = 0 \quad \text{or} \quad \dot{\vec{R}} = \text{constant}$$

we already knew this from conservation of the total momentum of the system.

we can also say $\vec{R}$ is "ignorable"

$\vec{r}$ equation

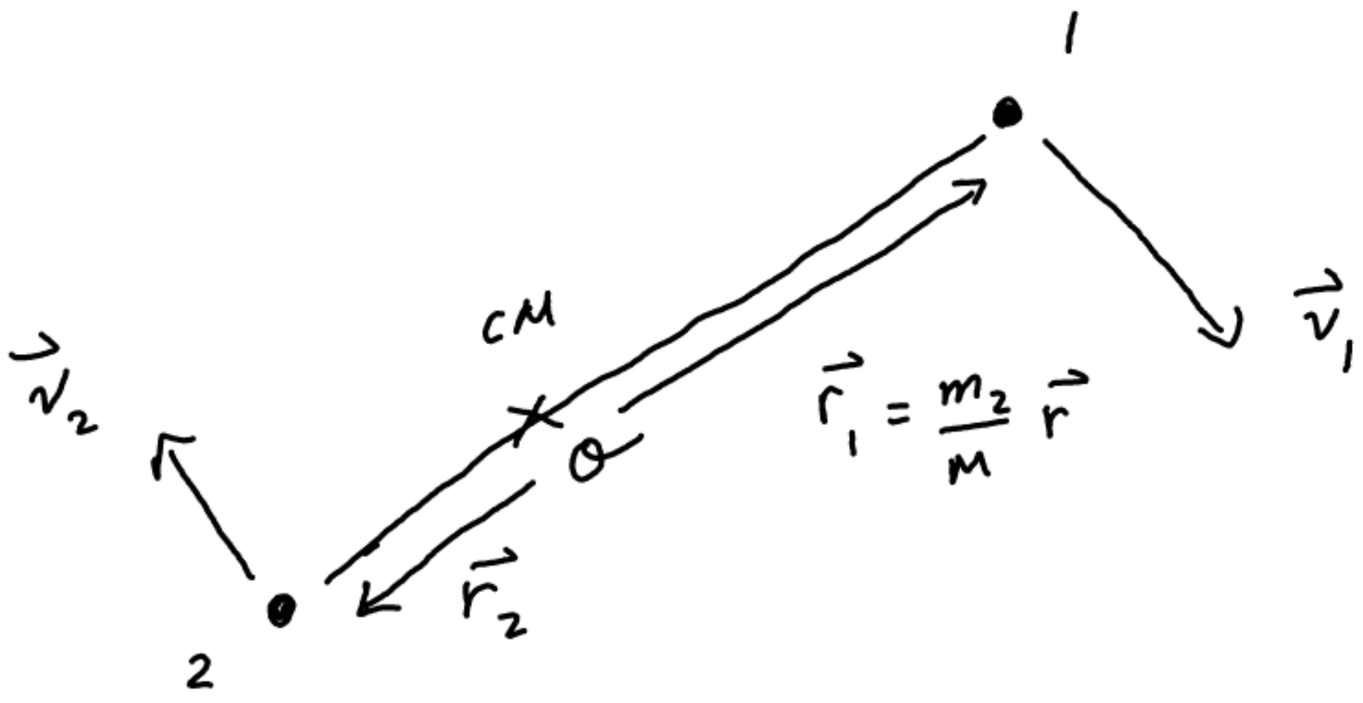
$$\frac{\partial \mathcal{L}}{\partial \vec{r}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = -\frac{\partial U}{\partial \vec{r}} - \mu \ddot{\vec{r}} = 0$$

$$\mu \ddot{\vec{r}} = -\nabla U(\vec{r})$$

like a single particle under the influence of some force.

Since $\vec{R}$ is an ignorable coordinate, we will work in the CM frame, where $\vec{R} = 0$.

$$\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(\vec{r})$$



Two bodies have equal, opposite momentum

CM is stationary at origin O

Bodies rotate CM

Two-dimensional problem

We know our orbit must stay in the 2-d plane because angular momentum is conserved.

Equations of motion in 2-d

Our 2-d, unconstrained problem described by position $\vec{r}$ can be broken up into coordinates $\vec{r} = (r, \varphi)$.

Then, $$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

notice $\mathcal{L}$ independent of φ $\rightarrow$ φ ignorable

(angular momentum conserved!)

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} = \text{const.} = l_z$$

radial equation

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\underbrace{\mu r \dot{\varphi}^2}_{\text{centripedal acceleration}} - \frac{dU}{dr} = \mu \ddot{r}$$

Equations of Motion 1-d

since $\dot{\varphi}$ is ignorable, we can eliminate it from the original Lagrangian:

$$\dot{\varphi} = \frac{l}{\mu r^2}$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - u(r)$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + \frac{l^2}{\mu^2 r^2}\right) - u(r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$+\frac{l^2}{\mu r^3} - \frac{du}{dr} = \mu\ddot{r}$$

"fictitious" centrifugal force ↑ actual force

$$-\frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right) - \frac{du}{dr} = \mu\ddot{r}$$

↑ centrifugal potential energy

Effective potential: $u_{eff} = u(r) + \frac{l^2}{2\mu r^2}$