Homework 7: Rigid Rotors, Taylor Ch. 10

1. For the following problems, *show your work* to get full credit, and no Mathematica allowed! (You should be able to do all of the below on an exam, where you won't have Mathematica.)

   Consider the following 3x3 matrices:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & -1 & 0 \end{pmatrix}$$

   (a) Find the eigenvalues of matrix A, and then the corresponding set of eigenvectors. Please clearly label which eigenvector goes with which eigenvalue, and normalize the eigenvectors to have unit length.

   (b) Find the eigenvalues of matrix B, and write down a set of **orthonormal** (mutually perpendicular and length 1) eigenvectors. You should find some ambiguity in choosing two of the eigenvectors, due to a degeneracy: please take one of them to be the unit vector $\hat{v} = (1, 0, 0)$.

   (c) As you can verify, the matrix C has zero determinant. *Without solving* the characteristic polynomial $\det(C - \lambda I) = 0$, can you say anything about the eigenvalues of $\det(C)$? Can you say anything about some of the eigenvectors, again without solving the characteristic equation?

2. A thin, flat, uniform metal triangle lies in the xy plane with its corners at (1,0,0), (0,1,0), and the origin. Its surface density (mass/area) is $\sigma = 24$. (Distances and masses are measured in unspecified units, and the number 24 was chosen to make the answer come out nicely.)

   (a) Find the triangle's inertia tensor.

   (b) What are its principle moments and the corresponding axes?

3. A *loaded die* is shaped like a cube of side length $2b$, but instead of constant density is weighted more heavily towards one face (and less heavily towards the opposing face),

$$\rho(z) = \rho\left(1 + \frac{z}{2b}\right)$$

in coordinates with $\hat{z}$ perpendicular to the weighted face and the geometric center of the cube at $z = 0$.

(a) Because of the non-uniform density, the center of mass of the loaded die is no longer at the exact geometric center of the cube. Starting in a coordinate system with origin at the geometric center of the cube (so $\rho(x, y, z)$ is exactly as given above), calculate the position $\vec{R}$ of the *new* center of mass.

(b) Calculate the inertia tensor $\overleftrightarrow{I}$ of the loaded die about its center of mass. (You may do this either by changing to coordinates about the center-of-mass and integrating directly, or by calculating $\overleftrightarrow{I}$ first about the geometric center, and then shifting it using the parallel axis theorem.)

(*Hint:* much like the cube in class, many of the components of $\overleftrightarrow{I}$ are equal to each other due to the symmetries of the problem; if you think carefully, you don't have to do too many integrals.)

(c) You are handed a pair of dice, and one of them is fair (i.e. a uniform cube) while the other is the loaded die we've been considering; you don't know which is which. One way to decide which is the fair die would be to roll them over and over, and keep track of the probability of rolling each of the numbers, but that will take a while. Using your knowledge of mechanics, can you propose a faster (non-destructive) way to distinguish the two dice?