Homework 7

1. Eigenvalues + Eigenvectors

a)
$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)^2 - 6\right] = 0$$

$$\lambda_1 = 2$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-y + 3z = 0$$
$$2y - z = 0$$

only soln
$y = z = 0$

$$e_1 = (1, 0, 0)$$

$$(1-\lambda)^2 - 6 = 0$$

$$1-\lambda = \pm\sqrt{6}$$

$$\lambda = 1 \pm \sqrt{6}$$

$$\lambda_2 = 1 + \sqrt{6}$$

$$\begin{pmatrix} 1-\sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 3 \\ 0 & 2 & -\sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x = 0$$

$$-\sqrt{6}\,y + 3z = 0$$

$$y = \frac{3}{\sqrt{6}} z = \frac{3\sqrt{6}}{6} z = \frac{1}{2}\sqrt{6}\, z$$

normaliz: $\sqrt{1 + \frac{6}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$

$$e_2 = \frac{2}{\sqrt{10}} \left(0, \frac{\sqrt{6}}{2}, 1\right)$$

$$= \left(0, \sqrt{3/5}, \sqrt{2/5}\right)$$

$$\lambda_3 = 1 - \sqrt{6}$$

$$\begin{pmatrix} 1+\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 3 \\ 0 & 2 & \sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x = 0$$

$$\sqrt{6}\,y + 3z = 0 \longrightarrow y = -\frac{\sqrt{6}}{2} z$$

$$e_3 = \frac{2}{\sqrt{10}} \left(0, -\frac{\sqrt{6}}{2}, 1 \right)$$

$$= \left(0, -\sqrt{3/5}, \sqrt{2/5} \right)$$

b) $$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$\lambda_1 = 3$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$-y - z = 0 \quad \rightarrow -y = z$$

$$e_1 = (0, -1, 1)\frac{1}{\sqrt{2}}$$

$$(2-\lambda)^2 - 1 = 0$$

$$2-\lambda = \pm 1$$

$$\lambda = 2 \pm 1$$

$\lambda_2 = 1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x = 0$

$y = z$

$e_2 = \frac{1}{\sqrt{2}} (0, 1, 1)$

$\lambda_3 = 3$ $\quad$ x can be anything

$y = -z$

$e_3 = (1, 0, 0)$

c) We do not need to solve $(C-\lambda) = 0$

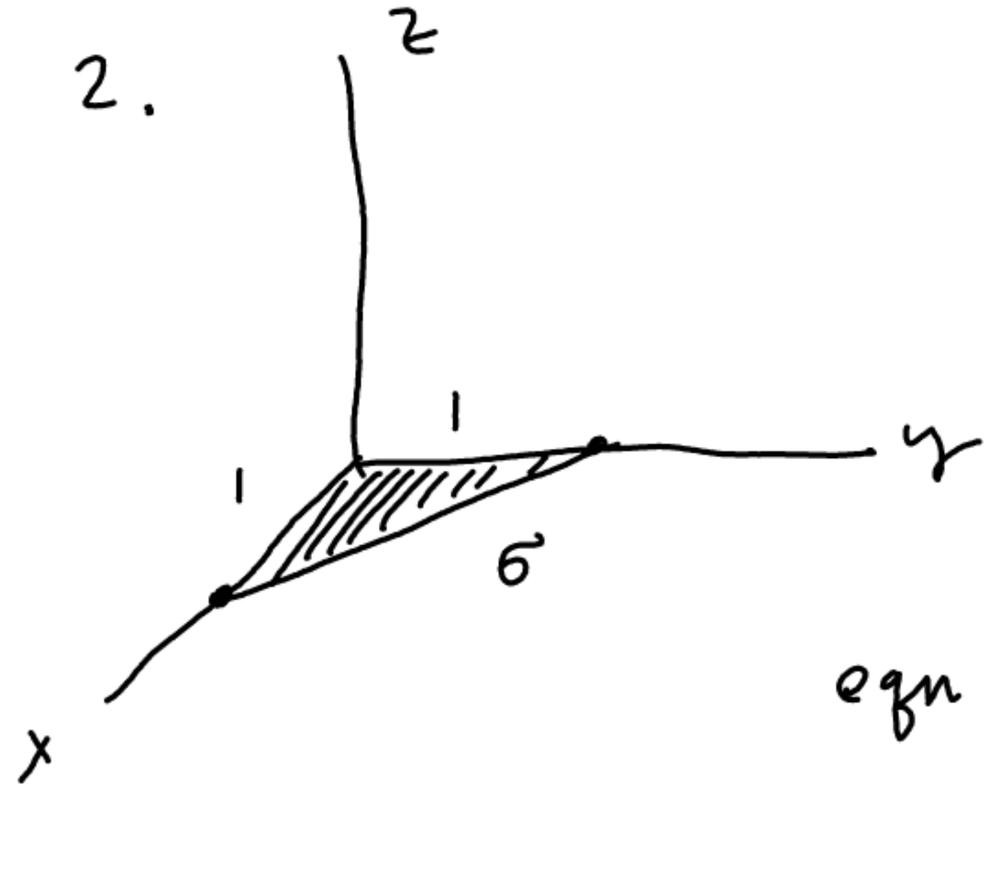
because we can immediately see that

the first two rows are <u>NOT</u> linearly independent.

One eigenvalue will be zero

and $\det C = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$

2.



$$I_{ij} = \int \sigma \, dA \; \delta_{ij} (x_i^2 + x_j^2) \cdot (1 - \delta_{ij}) \, x_i x_j$$

eqn of hypotenuse:

$$y = 1 - x$$

$$I_{xx} = \int_0^1 dx \int_0^{1-x} dy \;\; \sigma \;\; (y^2 + \cancel{z^2}^{\,0})$$

$$= \int_0^1 dx \;\; \sigma \left(\frac{1}{3} y^3 \right)_0^{1-x}$$

$$= \sigma \int_0^1 \frac{1}{3} (1-x)^3$$

$$u = 1 - x$$

$$du = -dx$$

$$= -\frac{\sigma}{3} \int_1^0 u^3 \, du = -\frac{\sigma}{3} \left(\frac{1}{4} u^4 \right)_1^0$$

$$I_{xx} = \frac{\sigma}{12}$$

$$I_{yy} = \int_0^1 dx \int_0^{1-x} dy\, 6\,(x^2 + z^2)$$

Should be the same as I_{xx}

$$I_{zz} = \int_0^1 dx \int_0^{1-x} dy\, 6\,(x^2 + y^2)$$

$$= 6 \int_0^1 dx \left(x^2 y + \tfrac{1}{3} y^3\right)_0^{1-x}$$

$$= 6 \int_0^1 dx \left[x^2(1-x) + \tfrac{1}{3}(1-x)^3\right]$$

$$= 6 \int_0^1 dx \left[x^2 - x^3 + \tfrac{1}{3}(1-x)^3\right]$$

$$= 6 \left[\int_0^1 dx\, x^2 - \int_0^1 dx\, x^3 + \frac{1}{3}\int_0^1 dx\,(1-x)^3\right]$$

$$= 6\left(\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{12}\right)$$

$$= \frac{6}{6}$$

$$I_{xy} = -\int_0^1 dx \int_0^{1-x} dy\, \sigma\, x\, y$$

$$= -\int_0^1 dx\; 6\; \tfrac{1}{2} x (1-x)^2 = -\int_0^1 dx\, \frac{6}{2} x(x^2 - 2x + 1)$$

$$= -\frac{6}{2}\int_0^1 (x^3 - 2x^2 + x) = -\frac{6}{2}\left(\tfrac{1}{4} - \tfrac{2}{3} + \tfrac{1}{2}\right)$$

$$I_{xy} = -\frac{6}{24}$$

$$I_{xz} = I_{yz} = 0 \qquad 6 = 24$$

$$\underline{\underline{I}} = 6 \begin{pmatrix} 1/12 & -1/24 & 0 \\ -1/24 & 1/12 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

b) $$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$\lambda_1 = 4$

$(2-\lambda)^2 - 1 = 0$

$2 - \lambda = \pm 1$

$\lambda = 2 \pm 1$

$\lambda_2 = 3 \qquad \lambda_3 = 1$

$\lambda_1 = 4$:

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x - y = 0$ $\quad$ z can be anything

$y = -2x$

$\lambda_2 = 3$:

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$z = 0$ $\quad$ $x = -y$

$\lambda_3 = 1$:

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

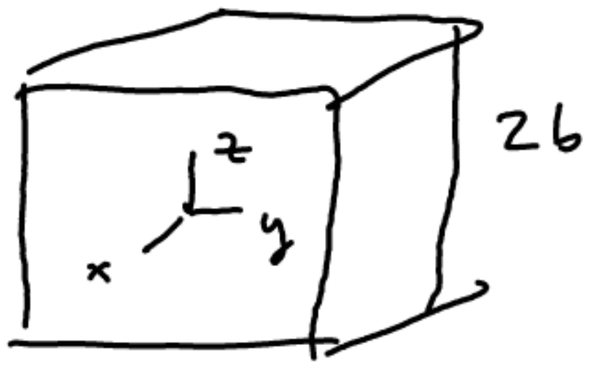
$z = 0$ $\quad$ $x = y$

$\lambda_1 = 4 \quad e_1 = (0, 0, 1)$

$\lambda_2 = 3 \quad e_2 = (1, -1, 0)\ \frac{1}{\sqrt{2}}$

$\lambda_3 = 1 \quad e_3 = (1, 1, 0)\ \frac{1}{\sqrt{2}}$

3.



$$\rho(z) = \rho\left(1 + \frac{z}{2b}\right)$$

a) $$\vec{R} = \frac{1}{M} \sum_i m_i r_i$$

$$= \frac{\int \vec{r}\,\rho(z)\,dV}{\int \rho(z)\,dV}$$

$$= \frac{\iiint (\vec{x} + \vec{y} + \vec{z})\,\rho\left(1 + \frac{z}{2b}\right)\,dx\,dy\,dz}{\iiint \rho\left(1 + z/2b\right)\,dx\,dy\,dz}$$

denominator:

$$M = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz\ \rho\left(1 + z/2b\right)$$

$$= \int_{-b}^{b} dx \int_{-b}^{b} dy\ \rho\left[z + \frac{1}{4b} z^2\right]_{-b}^{b}$$

$$= \int_{-b}^{b} dx \int_{-b}^{b} dy\ \rho\left[2b\right]$$

$$= \int_{-b}^{b} dx\ \rho \cdot 4b^2$$

$$M = 8\rho b^3$$

$$R_x = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \quad x\rho\left(1 + z/2b\right) \cdot \frac{1}{M}$$

when we integrate in x, $R_x \to 0$

same for R_y when we integrate in y

$$R_z = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \quad \rho z \left(1 + z/2b\right) \cdot \frac{1}{M}$$

$$= \int_{-b}^{b} dx \int_{-b}^{b} dy \; \rho \left[\cancel{\frac{1}{2} z^2}^{\,0} + \frac{1}{6b} z^3 \right]_{-b}^{b} \cdot \frac{1}{M}$$

$$= (2b)(2b) \cdot \frac{1}{3} b^2 \cdot \frac{1}{8b^3}$$

$$R_z = \frac{1}{6} b$$

$$\vec{R} = (0, \; 0, \; b/6)$$

b) $$I_{xx} = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \;\; \rho\left(1+\frac{z}{2b}\right)(y^2+z^2)$$

$$= (2b)\rho \int_{-b}^{b} dy \left[y^2 z + \frac{1}{3} z^3 \right]_{-b}^{b}$$

$$= (2b)\rho \int_{-b}^{b} dy \left[y^2(2b) + \frac{2}{3} b^3 \right]$$

$$= (2b)\rho \left[\frac{1}{3} y^3 (2b) + \frac{2}{3} b^3 y \right]_{-b}^{b}$$

$$= (2b)\cdot\rho\cdot\left(\frac{4}{3} b^4 + \frac{4}{3} b^4 \right)$$

$$= 16\rho\frac{b^5}{3} = \boxed{\frac{2}{3} M b^2}$$

$$I_{yy} = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \; \rho\left(1+\frac{z}{2b}\right)(x^2+z^2)$$

$$= \frac{2}{3} M b^2$$

int to zero

$$I_{zz} = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \; \rho\left(1+\cancel{\frac{z}{2b}}\right)(x^2+y^2)$$

$$= (2b)\rho \int_{-b}^{b} dx \left(x^2 y + \frac{1}{3} y^3 \right)_{-b}^{b}$$

z-int

$$= (2b)\rho \left[(2b)\left(\frac{2}{3} b^3\right) + \frac{2}{3} b^3 (2b) \right]$$

$$= \frac{2}{3} M b^2$$

$$I_{xy} = \int_{-b}^{b} dx \int_{-b}^{b} dy \int_{-b}^{b} dz \; \rho\left(1 + \frac{z}{2b}\right) x y$$

integrates to zero

So will I_{yz}, I_{xz}

$$\underline{\underline{I}} = \begin{pmatrix} \frac{2}{3}Mb^2 & 0 & 0 \\ 0 & \frac{2}{3}Mb^2 & 0 \\ 0 & 0 & \frac{2}{3}Mb^2 \end{pmatrix}$$ about $(0, 0, 0)$

For parallel axes, add

$$M\left(a^2 \delta_{ij} - a_i a_j\right)$$

$$\vec{a} = (0, 0, b/6)$$

all off-diagonal components are zero

$$xx = a^2 - a_z \cdot a_z = 0$$

$$zz = a^2 - 0 \cdot 0 = a^2 = b^2/36 = yy$$

$$\underline{\underline{I}}_{cm} - M\left(a^2 \delta_{ij} - a_{ij}\right)$$

$$= \begin{pmatrix} \frac{2}{3}Mb^2 - \frac{Mb^2}{36} & 0 & 0 \\ 0 & \frac{2}{3}Mb^2 - \frac{Mb^2}{36} & 0 \\ 0 & 0 & \frac{2}{3}Mb^2 \end{pmatrix}$$

c) Try to find CM by balancing on table edge

toss in air + watch for precession

$$(\lambda_3 \neq \lambda_2, \lambda_1)$$