Homework 6: Accelerating Reference Frames
Taylor Ch. 9

1. Rotating reference frames tutorial section III.

2. Consider the motion of a puck on a frictionless, perfectly flat record spinning counterclockwise with angular velocity Ω. In the non-inertial frame of reference rotating with the record, set up an x-y coordinate system. You can ignore the Earth's rotation in the following – only consider the rotation of the record.

   (a) Write down the x and y equations of motion for the puck on the record. Include both the centrifugal and Coriolis forces.

   (b) Solve the equations derived in part a by using the trick $\eta = x + iy$, and guessing a solution of the form $\eta = \exp(-i\alpha t)$. A second solution can also be written $\eta = t\exp(-i\alpha t)$.

   (c) At $t = 0$, the puck is pushed from $(x_0, 0)$ with initial velocity (v_{x0}, v_{y0}). Show that

$$x(t) = (x_0 + v_{x0}t)\cos(\Omega t) + (v_{y0} + \Omega x_0)t\sin(\Omega t) \tag{1}$$

$$y(t) = -(x_0 + v_{x0}t)\sin(\Omega t) + (v_{y0} + \Omega x_0)t\cos(\Omega t) \tag{2}$$

   (d) Sketch the solution for large times $t \to \infty$. What does the trajectory of the puck look like?

3. Magnitude of Coriolis and Centrifugal Forces on Earth

   On this problem, we will get some intuition about the magnitude of the Coriolis force by considering its influence on a ball dropped from the top of the campus Campanile. Noting that the Campanile is approximately 60 meters tall, roughly estimate *both* the ratio of the Coriolis acceleration to the acceleration of gravity *and* the ratio of the centrifugal acceleration to the acceleration of gravity just before the ball hits the ground. You may assume that the ball is dropped from rest, and that air resistance is negligible. Don't worry about the geometric correction due to the latitude of the Campanile for this part of the problem: we will evaluate this in detail on the next problem.

   Are these ratios large or small? Do we normally need to account for the Coriolis and the centrifugal forces for objects dropped from rest near the surface of the Earth? What might be an example where they do need to be included? Briefly justify your answers.

4. A Re-Imagined Galileo Experiment: Coriolis Deflection of a Falling Body

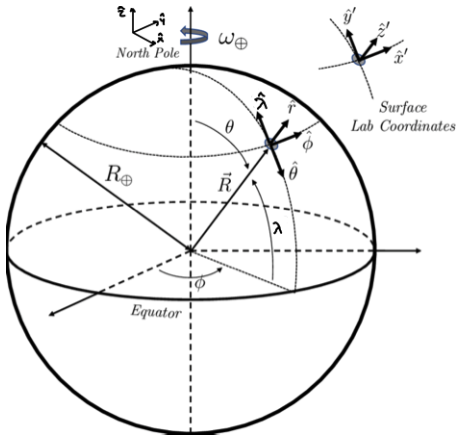


Figure 1: A schematic diagram depicting the coordinate systems used on this problem. The angular velocity of the Earth is aligned with the *global* z-axis, $\boldsymbol{\omega}_\oplus = \omega_\oplus \hat{\mathbf{z}}$. Also shown is the *local* spherical coordinate system $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ used in this problem. θ is the standard spherical polar angle, also referred to as the *colatitude* in geophysics and oceanography. For reference, a *local* Cartesian coordinate system $(\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$ is also shown. The local Cartesian coordinates $(\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$ can be thought of as pointing in the eastward, northward, and upward directions locally at a point on the surface of the Earth, respectively, and also aligned with the local spherical coordinates $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\lambda}})$, where λ is the *latitude*.

On the last problem, you should have found that the Coriolis deflection for a dropped object is typically very small. We will build from this to calculate the deflection for a ball dropped from the top of the Campanile in detail. Our key assumption is that the ball's deflection from the vertical is very small, so that its velocity can be approximated as purely vertical when evaluating the Coriolis acceleration. At the end of the problem, we will be able to answer the question: *Could Galileo have experimentally proven the rotation of the Earth by dropping a ball from the top of the Tower of Pisa?*[1]

We will work throughout this problem in *spherical coordinates* (r, θ, ϕ) with associated unit vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$, defined as follows:

- r: radial distance from Earth's center; $\hat{\mathbf{r}}$ points locally *upward* (radially outward).
- λ: latitude, the angle measured from the equator to the *north* pole; increasing λ points *north*; $\hat{\boldsymbol{\lambda}}$ lies in the local plane tangent to the surface (also referred to as the local horizontal plane) and points to the *north*.
- θ: colatitude, the angle measured downwards from the *north* pole; increasing θ points *southward*; $\hat{\boldsymbol{\theta}}$ is also in local horizontal plane, anti-aligned with $\hat{\boldsymbol{\lambda}}$, and points to the *south*.

[1] Historically, Isaac Newton suggested this very experiment to Robert Hooke, who attempted to measure it from the top of St. Paul's Cathedral in London in 1679-1680, as well as Giovanni Guglielmini in 1791 from the Asinelli Tower in Bologna, Italy. Both were unsuccessful due to the small size of the effect and experimental uncertainties. The first successful measurement was due to Johann Benzenberg in 1802, from the tower of St. Michael's Church in Hamburg, Germany– the first direct experimental evidence of the rotation of the Earth, though with considerable uncertainties. Foucault later established the rotation in a more convincing fashion with his famous pendulum in 1851 – a version of which you can see today in many museums across the world (see *List of Foucault pendulums*) – including the Museum of Science in Boston.

- ϕ: longitude; increasing ϕ points *eastward*; $\hat{\boldsymbol{\phi}}$ also lies in the horizontal plane, is tangent to the parallel (line of latitude), and points *east*.

Colatitude θ is the standard spherical coordinate, whereas latitude λ is usually used in geophysics and oceanography. The relationship between latitude (in radians) and colatitude is $\lambda = \dfrac{\pi}{2} - \theta$ (positive in the Northern Hemisphere).

We will make the simplifying assumptions that we can neglect air resistance, and take the height much less than the radius of the Earth $h \ll R_\oplus$, so that g is constant. The smallness of the Coriolis effect means that we can treat Coriolis effects as a small perturbation to the motion of the ball in pure vertical free fall. This means we are justified in computing the motion assuming purely vertical free fall, then evaluating the Coriolis acceleration $-2\boldsymbol{\omega} \times \mathbf{v}$ along that zeroth–order trajectory.

We sometimes say we are treating the Coriolis effect to "first order" in ω, treating ω as a "small parameter" when all other parameter are held fixed.

(a) A ball is released from rest at height h above the surface at latitude λ. Write the velocity $\mathbf{v}(t)$ of the ball in *local spherical* components assuming purely vertical motion (i.e. ignore Coriolis while determining $\mathbf{v}$). State clearly which component(s) are nonzero and give $\mathbf{v}(t)$ in terms of g and t.

(b) Let $\boldsymbol{\omega}$ be Earth's angular velocity vector, directed along its rotation axis $\hat{\mathbf{z}}$. Starting from the Cartesian representation

$$\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}},$$

express $\hat{\mathbf{z}}$ in terms of the *local* spherical basis at a point (r, θ, ϕ) by noting that the global Cartesian coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ are mapped to the local spherical vectors $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ by a rotational transformation $R(\theta, \phi)$ through the angles θ and ϕ:

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \underbrace{\begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}}_{R(\theta,\phi)} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix}.$$

Hence derive $\boldsymbol{\omega}$ in spherical components of the form

$$\boldsymbol{\omega} = \omega\big(A(\theta)\,\hat{\mathbf{r}} + B(\theta)\,\hat{\boldsymbol{\theta}}\big),$$

and identify $A(\theta)$ and $B(\theta)$.

(c) Using $\lambda = \dfrac{\pi}{2} - \theta$, rewrite your result from part (b) so that $\boldsymbol{\omega}$ is expressed in terms of λ, i.e.

$$\boldsymbol{\omega} = \omega\big(C(\lambda)\,\hat{\mathbf{r}} + D(\lambda)\,\hat{\boldsymbol{\theta}}\big),$$

and identify $C(\lambda)$ and $D(\lambda)$. (Be careful with the sign of the $\hat{\boldsymbol{\theta}}$ component: $\hat{\boldsymbol{\theta}}$ points *south*.)

(d) Using your velocity from part (a) and $\boldsymbol{\omega}$ from part (c), compute the Coriolis acceleration

$$\mathbf{a}_C(t) = -\,2\,\boldsymbol{\omega} \times \mathbf{v}(t).$$

Work to first order in ω (i.e. evaluate $\mathbf{a}_C$ along the zeroth–order vertical trajectory), and show which horizontal component(s) of $\mathbf{a}_C$ are nonzero. Integrate in time *to obtain the net horizontal displacement at impact* (express your answer in the local spherical coordinates using $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$, and/or $\hat{\boldsymbol{\theta}}$), in closed form as a function of λ, h, g, and ω.

Hint: It is easiest to evaluate the cross product using component form, in terms of the determinant:

$$\boldsymbol{\omega} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ \omega_r & \omega_\theta & \omega_\phi \\ v_r & v_\theta & v_\phi \end{vmatrix} = (\omega_\theta v_\phi - \omega_\phi v_\theta)\,\hat{\mathbf{r}} + (\omega_\phi v_r - \omega_r v_\phi)\,\hat{\boldsymbol{\theta}} + (\omega_r v_\theta - \omega_\theta v_r)\,\hat{\boldsymbol{\phi}}.$$

If $t_f = \sqrt{2h/g}$ is the fall time, you will find a displacement of the form

$$\Delta x(\lambda) \propto \omega\; g\; t_f^3 \cos(\lambda). \tag{3}$$

This result, which is now a homework exericse for PHY 313, is identical to the result derived by the giants of physics Laplace and Gauss for the first time in 1803 – and over a century from the publication of Newton's *Principia* in 1687!

(e) Evaluate your symbolic result from part (d) numerically for a heavy ball dropped from the top of the Campanile at a height of 60 m. Take

$$\lambda \approx 41.6^\circ \;\; \text{(North Dartmouth, MA)}, \quad g = 9.81\ \mathrm{m\,s^{-2}}, \quad \omega = 7.292 \times 10^{-5}\ \mathrm{s^{-1}},$$[2]

Report the magnitude of the final displacement from the undeflected impact point in centimeters and state its compass direction (north, east, south, or west) relative to the point directly beneath the release point. Briefly justify the direction you find based on the sign of the relevant component(s) of $\mathbf{a}_C$.

Could a modern day Galileo "discover" the rotation of the Earth by measuring its Coriolis deflection by dropping a ball from the top of the Campanile? For fun (not required), what would the deflection be from the world's tallest building – the Burj Khalifa in Dubai ($h = 828$ m, $\lambda \approx 25.2^\circ$)?