Homework 4: Two-Body Central Force Problems
Taylor Ch. 8

1. A satellite of mass m is in the vicinity of a planet of mass M. We'll study the motion in the plane containing the planet and the path of the satellite, so we can treat the problem as two-dimensional.

   (a) Look back at the results of our derivations of the equations of motion for a general central force problem. Replace the general form $U(r)$ with the gravitational potential energy and show that there is one equilibrium value of r, corresponding to a circular orbit.

   (b) A *geostationary* orbit is a particular circular orbit in which a satellite remains fixed in position above a spot above the planet's surface, i.e. the motion of the satellite satisfies the conditions $\ddot{r} = 0$ and $\dot{\theta} = \Omega$, where Ω is the angular speed of the planet's rotation. Find the fixed radius R at which a geostationary orbit is possible; it should depend only on the gravitation of the planet, i.e. on G, M, m, and Ω. Then take $M >> m$ to simplify dependence to only G, M, and Ω.

   Now suppose the satellite is perturbed from its geostationary orbit by a small amount, i.e. take $r = R + \epsilon$. Taylor expand the acceleration to first order in ϵ, and show that the geostationary orbit is a point of *stable equilibrium*. What is the frequency ω of small oscillations in r?

   (Hint: here is a helpful calculus identity: $\dfrac{1}{(1+x)^n} = 1 - nx + (...)$ *for small* x*, where* $(...)$ *indicates terms of order* x^2 *and higher.*)

   (c) The presence of a small oscillation in r will induce fluctuations in the previously constant angular speed $\dot{\theta}$ as well. Again working to first order in ϵ, solve for the modified $\theta(t)$. The orbital period T is the time required for θ to change by 2π; show that even with the added ϵ term, the orbital period is unchanged to this order.

2. An earth satellite is observed at perigee to be 250 km above the earth's surface and traveling at about 8500 m/s. Find the eccentricity of its orbit and its height above the earth at apogee. [Hint: The earth's radius is $R_e \sim 6.4 \times 10^6$ m. You will also need to know GM_e, but you can find this if you remember that $GM_e/R_e^2 = g$.]

3. The *Nemesis hypothesis* is a non-terrestrial attempt to explain an apparent cycle in the record of great extinctions on the Earth of approximately 27 million years. Nemesis is the name of a hypothetical, very dim star orbiting the Sun at a very great distance, so that its orbit passes through the Oort cloud, an enormous cloud of comets surrounding the solar system. Its regular passage through the Oort cloud would cause a number of comets to bombard the inner solar system, triggering extinction events when the Earth is impacted [1].

   If Nemesis exists, it is most likely a very dim dwarf star or even a brown dwarf (think Jupiter but bigger), so we will **assume that the mass of Nemesis is much less than the mass of the Sun** in all of the below.

   (a) Given an orbital period of $\tau = 27$ Myr, estimate the semimajor axis a of the orbit of Nemesis. Assuming the outer edge of the Oort cloud is at approximately 50,000 AU from the Sun, what is the minimum eccentricity ϵ that Nemesis must have in order to pass through the cloud? (One AU $\approx 1.5 \times 10^8$ km.)

   (b) The area of an ellipse is equal to πab, where a and b are the lengths of the semimajor and semiminor axes respectively. What is the area swept out by the orbit of Nemesis based on the orbital parameters you found? (Use the minimum eccentricity.) How many times would the state of Colorado (area $270,000\text{km}^2$) fit into this area?

   (c) The last major extinction event in the fossil record was approximately 11 million years ago. If Nemesis was at perihelion 11 million years ago, and its orbital eccentricity is equal to what you calculated in part A, how long will it take to reach aphelion, and what distance will it be from the Sun then?

[1] Both the existence of Nemesis and the periodicity of extinction events are actively debated, and are disfavored by current evidence, but the hypothesis still makes for a good physics problem!