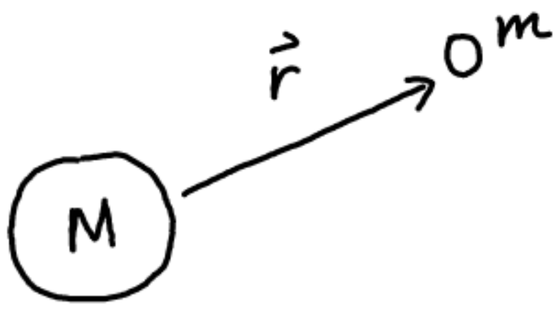
Homework 4

1.



a) We found the 1-d equation of motion:

$$+\frac{l^2}{\mu r^3} - \frac{dU}{dr} = \mu \ddot{r}$$

$$U_g(r) = -\frac{GMm}{r}$$

$$\mu \ddot{r} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) + \frac{l^2}{\mu r^3}$$

$$\mu \ddot{r} = -\frac{GMm}{r^2} + \frac{l^2}{\mu r^3}$$

equilibrium: $\ddot{r} = 0$ for $\dot{r}(0) = 0$

$$0 = -\frac{GMm}{r_0^2} + \frac{l^2}{\mu r_0^3}$$

$$GMm \cdot \mu r_0^{\not 3} = \not{r_0^2} l^2$$

$$r_0 = \frac{l^2}{GMm\mu} \quad \leftarrow \text{constant!}$$

a circle

b) $\ddot{r} = 0$ and $\dot{\varphi} = \Omega$ for some R

$$l = \mu r^2 \dot{\varphi} = \mu R^2 \Omega$$

$$r_0 = \frac{l^2}{GMm\mu}$$

$$\downarrow$$

$$R = \frac{(\mu R^2 \Omega)^2}{G \mu M m} = \frac{\mu^2 R^4 \Omega^2}{G \mu M m}$$

$$GMm = \mu R^3 \Omega^2$$

$$R^3 = \frac{GMm}{\mu \Omega^2} = \frac{G(M+m)}{\Omega^2}$$

now let $r = R + \varepsilon$

$$\mu \ddot{r} = -\frac{GMm}{r^2} + \frac{l^2}{\mu r^3}$$

$$\ddot{\varepsilon} = -\frac{GMm}{(R+\varepsilon)^2} + \frac{l^2}{\mu (R+\varepsilon)^3}$$

$$= -\frac{GMm}{R^2(1+\varepsilon/R)^2} + \frac{l^2}{\mu R^3 (1+\varepsilon/R)^3}$$

$$= -\frac{GMm}{R^2}(1-2\varepsilon/R) + \frac{l^2}{\mu R^3}(1-3\varepsilon/R)$$

$l = \mu R^2 \Omega$ $\qquad R^3 = \frac{GMm}{\mu \Omega^2}$

$GMm = R^3 \mu \Omega^2$

$$= -\frac{R^3 \mu \Omega^2}{R^2}(1-2\varepsilon/R) + \frac{\mu^2 R^4 \Omega^2}{\mu R^3}(1-3\varepsilon/R)$$

$$= -\mu \Omega^2 R (1-2\varepsilon/R) + \mu \Omega^2 R (1-3\varepsilon/R)$$

$$= \mu \Omega^2 R \left(\frac{2\varepsilon}{R} - \frac{3\varepsilon}{R}\right)$$

$\ddot{\varepsilon} = -\mu \Omega^2 \varepsilon$ an oscillation with $\omega = \sqrt{\mu \Omega^2}$

c) $\dot{\varphi} = \dfrac{l}{\mu r^2}$ $\qquad r = R + \varepsilon$

$$\dot{\varphi} = \frac{l}{\mu (R+\varepsilon)^2} = \frac{l}{\mu R \left(1 + \varepsilon/R\right)^2}$$

$$\dot{\varphi} = \frac{l}{\mu R}\left(1 - 2\varepsilon/R\right)$$

$$\int_0^{\varphi(t)} d\varphi = \frac{l}{\mu R}\int_0^t \left(1 - 2\varepsilon/R\right) dt$$

$\varepsilon = \varepsilon_0 \cos(wt)$

$w = \sqrt{\mu}\,\Omega$

$$\varphi(t) = \frac{l}{\mu R}\left(t - \frac{2}{R}\frac{\varepsilon_0}{w}\sin(wt)\right)$$

let $\tau = \dfrac{2\pi}{\Omega}$ be the period.

$l = \mu R^2 \Omega$, $w^2 = \mu \Omega^2$

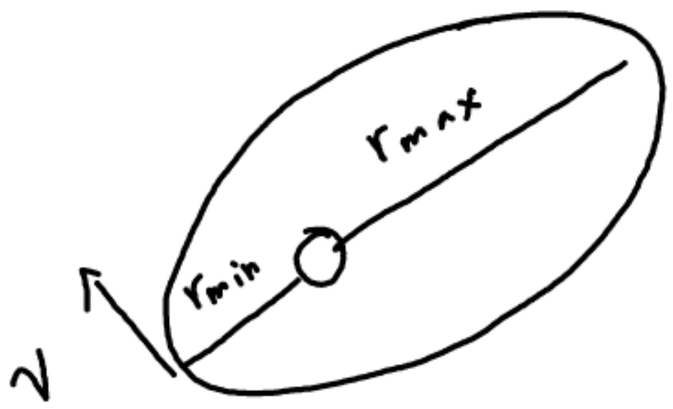
$$\varphi\left(\frac{2\pi}{\Omega}\right) = \frac{\mu R^2 \Omega}{\mu R}\left(\frac{2\pi}{\Omega} - \frac{2}{R}\frac{\varepsilon_0}{\sqrt{\mu}\,\Omega}\sin\left(\sqrt{\mu}\,\Omega \cdot \frac{2\pi}{\Omega}\right)\right)$$

for $M >> m$, $\mu \to 1$

$\sin 2\pi \to 0$

orbit period does not change.

2.



$$r_{min} = 250\ km + R_E$$

$$v = 8500\ m/s$$

$$r_{min} = \frac{c}{1+\varepsilon}$$

$$r_{max} = \frac{c}{1-\varepsilon}$$

Find ε:

$$c = r_{min}(1+\varepsilon)$$

$$v(\varphi) = \sqrt{\frac{GM}{c}}\sqrt{\frac{c^2}{r^2} + \varepsilon^2 \sin^2\varphi}$$

$$v_p = \sqrt{\frac{GM}{r_{min}(1+\varepsilon)}}\sqrt{\frac{r_{min}^2(1+\varepsilon)^2}{r^2} + \varepsilon^2\sin^2\varphi}$$

$$v_p^2 = \frac{GM}{r_{min}} \cdot \left[(1+\varepsilon) + \frac{\varepsilon^2}{1+\varepsilon}\sin^2\varphi\right]^{\varphi = 0}$$

$$\varepsilon = \frac{v_p^2 r_{min}}{GM} - 1 \quad \leftarrow \quad \frac{GM}{R_E^2} = g$$

$$GM = gR_E^2$$

$$\varepsilon = \frac{v_p^2 r_{min}}{gR_E^2} - 1 = 0.197$$

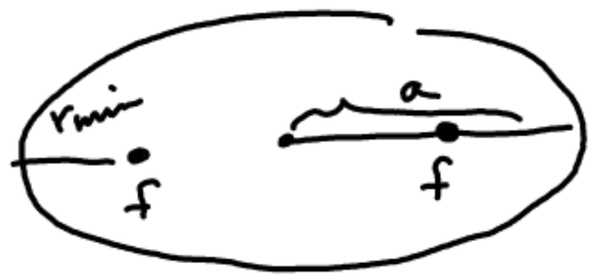
$$r_{max} = r_{min} \cdot \frac{1+\varepsilon}{1-\varepsilon} = 9913\ km \rightarrow h_{max} = r_{max} - R_E$$

$$= \boxed{3513\ km}$$

3. $m_1 << m_2 \rightarrow \gamma \sim GM_\mu$

a) $\tau = 27 \times 10^6$ years find a

$$\tau^2 \sim \frac{4\pi^2}{GM} a^3$$



$$a^3 \sim \frac{\tau^2 GM}{4\pi^2}$$

$$= 1.35 \times 10^{16} \text{ m}$$

$$= 90{,}000 \text{ A.U.}$$

Oort cloud at 50,000 A.U. Less than a.

So r_{min} should be $\sim$ 50,000 A.U. in order to pass through the cloud.

$$r_{min} = \frac{c}{1+\varepsilon} \qquad a = \frac{c}{1-\varepsilon^2} \rightarrow c = a(1-\varepsilon^2)$$

$$r_{min} = a(1-\varepsilon)(1+\varepsilon)/(1+\varepsilon)$$

$$r_{min} = a(1-\varepsilon)$$

$$\varepsilon = -\frac{r_{min}}{a} + 1 = -\frac{50{,}000 \text{ A.U.}}{90{,}000 \text{ A.U.}} + 1 = 4/9 \sim 0.44$$

b) $\frac{b}{a} = \sqrt{1-\varepsilon^2} \rightarrow b = a\sqrt{1-\varepsilon^2}$

$\pi a b = \pi a^2 \sqrt{1-\varepsilon^2}$

$= \pi \cdot (90000 \text{ A.u.} \times 1.5 \times 10^8 \text{ km/A.u.})^2$

$\times \sqrt{1-(4/9)^2}$

$= 2.28 \times 10^{10} \text{ A.u.}^2$

$= \boxed{5.13 \times 10^{26} \text{ km}^2}$

That's 1.8×10^{21} Colonados.

(2 trillion)

c) time $= \tau/2 = \frac{27 \text{ Myr}}{2} = \boxed{13.5 \text{ Myr}}$

So 2.5 Myr from now.

$r_{max} = r_{min} \cdot \frac{1+\varepsilon}{1-\varepsilon} = 50{,}000 \text{ A.u.} \times \frac{13/9}{5/9}$

$= \boxed{130{,}000 \text{ A.u.}}$