Homework 3: Lagrangian Mechanics, Taylor Ch. 7

1. The center of a long frictionless rod is pivoted at the origin, and the rod is forced to rotate in a horizontal plane with constant angular velocity ω. Write down the Lagrangian for a bead of mass m threaded on the rod, using r as your generalized coordinate, where r, ϕ are the polar coordinates of the bead. (Notice that ϕ is not an independent variable since it is fixed by the rotation of the rod to be $\phi = \omega t$.) Solve Lagrange's equation for $r(t)$. What happens if the bead is initially at rest at the origin? If it is released from any point $r_0 > 0$, show that $r(t)$ eventually grows exponentially.

2. A particle is confined to move on the surface of a circular cone with its axis on the vertical z-axis, vertex at the origin (pointing down), and half-angle α. Gravity acts on the particle.

   (a) Write down the Lagrangian $\mathcal{L}$ in terms of the spherical polar coordinates r and ϕ.

   (b) Find the two equations of motion. Interpret the ϕ equation in terms of the angular momentum ℓ_z, and use it to eliminate $\dot{\phi}$ from the r equation in favor of the constant ℓ_z. Does your r equation make sense in the case that $\ell_z = 0$? Find the value r_0 of r at which the particle can remain in a horizontal circular path.

   (c) Suppose that the particle is given a small radial kick, so that $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is small. Use the r equation to decide whether the circular path is stable. If so, with what frequency does r oscillate about r_0?

3. Consider a bead of mass m sliding without friction on a wire that is bent in the shape of a parabola and is being spun with constant angular velocity ω about its vertical axis, as shown in the figure. Gravity acts on the bead. Use cylindrical polar coordinates and let the equation of the parabola be $z = k\rho^2$. Write down the Lagrangian in terms of ρ as the generalized coordinate. Find the equation of motion of the bead and determine whether there are positions of equilibrium, that is, values of ρ at which the bead can remain fixed, without sliding up or down the spinning wire. Discuss the stability of any equilibrium positions you find.