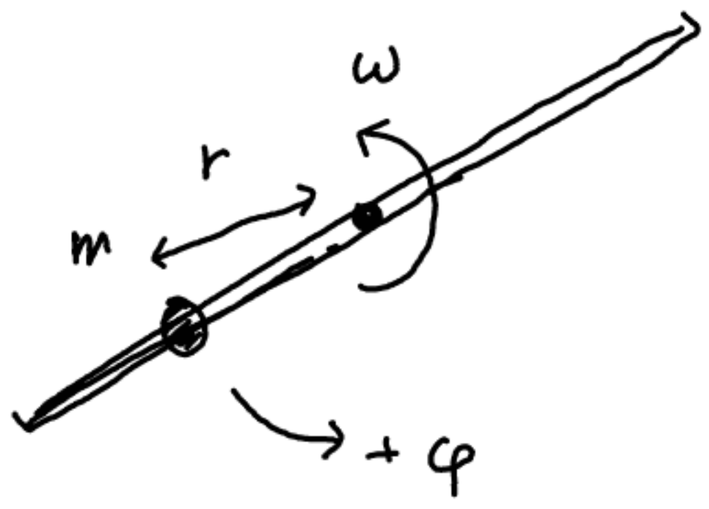
Homework 3: Lagrangian Mechanics

Hard Problems

1.



$$T = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2$$

$$T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 w^2$$

Since the whole system sits on a horizontal plane,

$$\mathcal{U} = 0$$

$$\mathcal{L} = T - \mathcal{U} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 w^2$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r w^2 \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}[m\dot{r}] = m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \longrightarrow \ddot{r} = r w^2$$

Solved by an exponential

$$r(t) = A e^{wt} + B e^{-wt}$$

A and B depend on initial conditions.

If the bead is initially at rest at the origin,

$$r(0) = A e^{w(0)} + B e^{-w(0)} = A + B = 0$$

$$A = -B$$

$$\dot{r}(t) = Awe^{wt} - Bwe^{-wt}$$

$$r(0) = Aw - Bw = 0 \longrightarrow A = B$$

only possible if $A = B = 0$

$r(t) = 0$ for all time

If the bead has $r(0) = r_0$ small,

$r(0) = r_0 = A + B$

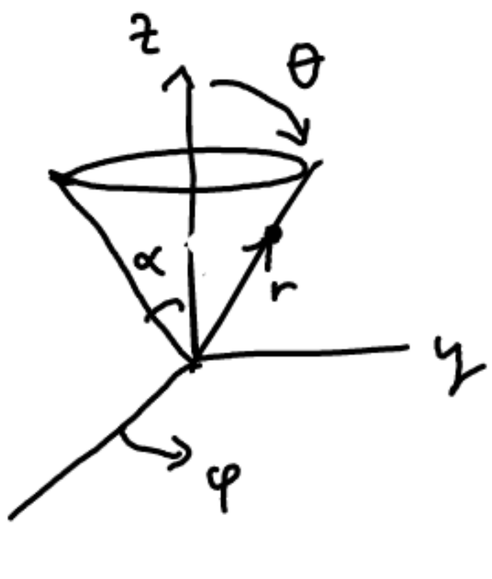
$\dot{r}(0) = 0$ still gives $A = B$

$$A = B = r_0/2$$

$$r(t) = \frac{r_0}{2} e^{wt} + \frac{r_0}{2} e^{-wt}$$



2.

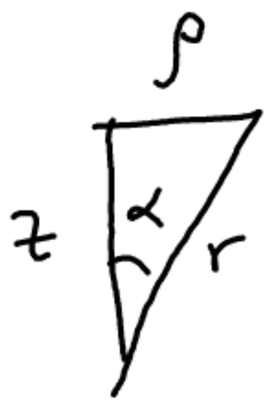


a) $T = \frac{1}{2} m v^2$

$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

Use spherical polar coordinates

θ is always fixed at $\theta = \alpha$



$x = \rho \cos\varphi$

$= r \sin\alpha \cos\varphi \rightarrow \dot{x} = \dot{r} \sin\alpha \cos\varphi - r \sin\alpha \sin\varphi \dot{\varphi}$

$y = r \sin\alpha \sin\varphi \rightarrow \dot{y} = \dot{r} \sin\alpha \sin\varphi + r \sin\alpha \cos\varphi \dot{\varphi}$

$z = r \cos\alpha \longrightarrow \dot{z} = \dot{r} \cos\alpha$

$T = \frac{1}{2} m \left(\dot{r}^2 \sin^2\alpha + r^2 \sin^2\alpha \dot{\varphi}^2 + \dot{r}^2 \cos^2\alpha \right)$

$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \dot{\varphi}^2 \right)$

$U = mgz = mgr\cos\alpha$

$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \sin^2\alpha \dot{\varphi}^2 \right) - mgr\cos\alpha$

b) $\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\alpha \, \dot{\varphi}^2) - mgr\cos\alpha$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m r \sin^2\alpha \, \dot{\varphi}^2 - mg\cos\alpha - \frac{d}{dt}\left[m\dot{r} \right] = 0$$

$$m r \sin^2\alpha \, \dot{\varphi}^2 - mg\cos\alpha = m\ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0 - \frac{d}{dt}\left[m r^2 \sin^2\alpha \, \dot{\varphi} \right] = 0$$

$$l_z = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \text{const.}$$

putting this back into the r- eqn,

$$l_z^2 = m^2 r^4 \sin^4\alpha \, \dot{\varphi}^2$$

$$\frac{l_z^2}{m r^3 \sin^2\alpha} - mg\cos\alpha = m\ddot{r}$$

If $l_z = 0$, $\ddot{r} = -g\cos\alpha$ like sliding down a ramp.

To get a horizontal circle, $\ddot{r} = 0$

$$\frac{l_z^2}{m r_0^3 \sin^2\alpha} = mg\cos\alpha \longrightarrow r_0 = \left[\frac{l_z^2}{m^2 g \sin^2\alpha \cos\alpha} \right]^{1/3}$$

c) $r(t) = r_0 + \varepsilon(t)$

$$m\ddot{r} = \frac{l_z^2}{m r^3 \sin^2\alpha} - mg\cos\alpha$$

$$\ddot{r} = \frac{l_z^2}{m^2 r^3 \sin^2\alpha} - g\cos\alpha$$

$$\ddot{\varepsilon} = \frac{l_z^2}{m^2 (r_0 + \varepsilon)^3 \sin^2\alpha} - g\cos\alpha$$

$$= \frac{l_z^2}{m^2 \sin^2\alpha} \frac{1}{r_0^3 (1 + \varepsilon/r_0)^3} - g\cos\alpha$$

$$= \frac{l_z^2}{m^2 r_0^3 \sin^2\alpha} (1 + \varepsilon/r_0)^{-3} - g\cos\alpha$$

take ε small : $(1 + \varepsilon/r_0)^{-3} = 1 - 3\varepsilon/r_0$

$$\ddot{\varepsilon} = \frac{l_z}{m^2 r_0^3 \sin^2\alpha} (1 - 3\varepsilon/r_0) - g\cos\alpha$$

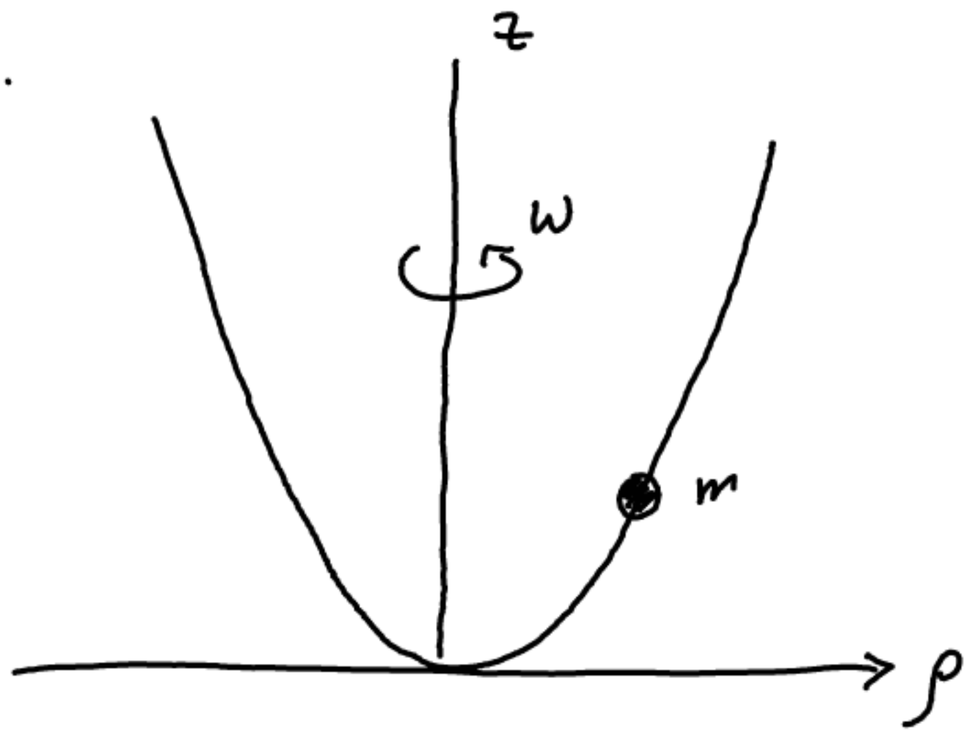
since r_0 is the equilibrium position, we can say r_0 is such that the first and last terms cancel.

$$\ddot{\varepsilon} = \frac{-3 l_z^2}{m^2 r_0^4 \sin^2\alpha}\,\varepsilon$$

an oscillation with frequency

$$\omega^2 = \frac{3 l_z^2}{m^2 r_0^4 \sin^2\alpha}$$

3.



$$z = k\rho^2$$

cylindrical coordinates:

$$x = \rho\cos\varphi \rightarrow \dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}$$

$$y = \rho\sin\varphi \rightarrow \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi}$$

constraint:

$$z = k\rho^2 \longrightarrow \dot{z} = 2k\rho\dot{\rho}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m(\dot{\rho}^2\cos^2\varphi + \rho^2\sin^2\varphi\,\dot{\varphi}^2 - 2\rho\dot{\rho}\cos\varphi\sin\varphi\,\dot{\varphi}$$
$$+ \dot{\rho}^2\sin^2\varphi + \rho^2\cos^2\varphi\,\dot{\varphi}^2 + 2\rho\dot{\rho}\cos\varphi\sin\varphi\,\dot{\varphi}$$
$$+ 4k^2\rho^2\dot{\rho}^2)$$

ω

$$T = \frac{1}{2}m(\dot{\rho}^2(1 + 4k^2\rho^2) + \rho^2\dot{\varphi}^2)$$

$$U = mgz = mgk\rho^2$$

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{\rho}^2(1+4k^2\rho^2) + \rho^2\omega^2) + mgk\rho^2$$

$$\mathcal{L} = T - U = \frac{1}{2} m \left(\dot{\rho}^2 \left(1 + 4k^2 \rho^2 \right) + \rho^2 \omega^2 \right) - mgk\rho^2$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = m\rho \left(4k^2 \dot{\rho}^2 + \omega^2 - 2gk \right)$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\rho}} = \frac{d}{dt} \left[m\dot{\rho} \left(1 + 4k^2 \rho^2 \right) \right]$$

$$= m\ddot{\rho} \left(1 + 4k^2 \rho^2 \right) + m\dot{\rho} \left(8k^2 \rho \dot{\rho} \right)$$

$$4k^2 \rho \dot{\rho}^2 + \rho \omega^2 - 2\rho g k$$

$$- \ddot{\rho} \left(1 + 4k^2 \rho^2 \right) + 8k^2 \rho \dot{\rho}^2 = 0$$

$$\left(1 + 4k^2 \rho^2 \right) \ddot{\rho} + 12k^2 \rho \dot{\rho}^2 + \rho \left(\omega^2 - 2gk \right) = 0$$

Equilibrium points ρ_0 have $\ddot{\rho} = 0$ if $\dot{\rho}(0) = 0$

So we can ignore the middle term ($\dot{\rho} = 0$)

$$\ddot{\rho} = \rho \cdot \frac{\left(\omega^2 - 2gk \right)}{1 + 4k^2 \rho^2}$$

$$\ddot{\rho} = \rho \frac{(w^2 - 2gk)}{1 + 4k^2\rho^2}$$

small ρ : $\frac{1}{1+4k^2\rho^2} \sim 1$

$$\ddot{\rho} = \rho\,(w^2 - 2gk)$$

$\rho_0 \neq 0$ {

$w^2 - 2gk < 0$ this is an SHO (stable)

$w^2 > 2gk > 0$ exponential (unstable)

If $w^2 = 2gk$, $\ddot{\rho} = 0$ for <u>any</u> ρ_0.