HW 2 : Lagrangian Mechanics

1. $F = -kx$, 1-D

$T = \frac{1}{2} m \dot{x}^2$ $\quad U = -\int \vec{F} \cdot d\vec{x}$ but I won't need to calculate this.

let $U = U(x)$

$\mathcal{L} = T - U = \frac{1}{2} m \dot{x}^2 - U(x)$

$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x = -kx$

$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$

$\rightarrow \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = -kx - m\ddot{x} = 0$

$m\ddot{x} = -kx$

as expected!

$\ddot{x} = -\frac{k}{m} x$

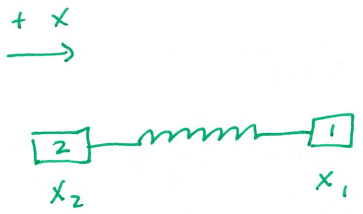
is solved by $x(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

if we take $x(0) = x_0$

$v(0) = 0$

then $x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

2.



Two particles in 1-D

Two d.o.f. → two coordinates

a) $T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$

$U = \frac{1}{2} k x^2 = \frac{1}{2} k (x_1 - x_2 - l)^2$

$$\mathcal{L} = T - U = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} k (x_1 - x_2 - l)^2$$

b) Let $X = \frac{1}{2}(x_1 + x_2)$, $x = (x_1 - x_2 - l)$

$\dot{X} = \frac{1}{2}(\dot{x}_1 + \dot{x}_2)$ $\dot{x} = \dot{x}_1 - \dot{x}_2$

$(2\dot{X})^2 = \dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_2$ $\dot{x}^2 = \dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}_1\dot{x}_2$

$$\dot{x}_1^2 + \dot{x}_2^2 = \frac{1}{2}\left[(2\dot{X})^2 + \dot{x}^2\right]$$

$$\mathcal{L} = \frac{1}{4} m \left[(2\dot{X})^2 + \dot{x}^2\right] + \frac{1}{2} k x^2$$

$$\mathcal{L} = m\dot{X}^2 + \frac{1}{4} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$+\frac{d}{dt}\left[2m\dot{X}\right] = 0 \qquad kx - \frac{d}{dt}\left[\frac{1}{2} m \dot{x}\right] = 0$$

c) $2m\ddot{\bar{X}} = 0$

$\dot{X} = \text{const} = V_0$

$X(t) = V_0 t + X_0$

the center of mass moves with constant speed.

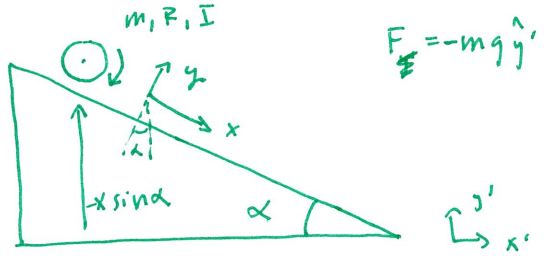
$kx - \frac{1}{2} m\ddot{x} = 0$

$\ddot{x} = \frac{2k}{m} x$

$x(t) = x_0 \cos\left(\sqrt{\frac{2k}{m}}\, t - \delta\right)$

The masses oscillate around relative to each other w/ frequency given by spring $2k$.

3.



$\vec{F} = -mg\hat{y}' = -mg\sin\alpha\,\hat{y}$

constraint: rolls without slipping

$v = R\omega$

$\dot{x} = R\dot{\theta}$

$\dot{\theta} = \dot{x}/R$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\dot{x}/R\right)^2$$

$$U = U(x,y) = mgh = -mgx\sin\alpha$$

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\dot{x}/R\right)^2 + mgx\sin\alpha$$

$$\mathcal{L} = \frac{1}{2}\left(m + I/R^2\right)\dot{x}^2 + mgx\sin\alpha$$

$$\frac{\partial\mathcal{L}}{\partial x} = +mg\sin\alpha \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \left(m + I/R^2\right)\dot{x}$$

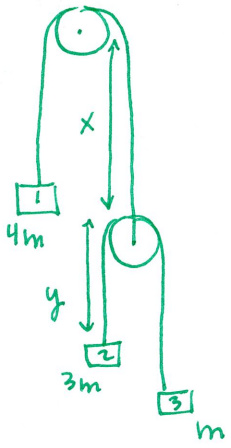
$$\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = +mg\sin\alpha - \left(m + I/R^2\right)\ddot{x} = 0$$

$$\ddot{x} = \frac{+mg\sin\alpha}{\left(m + I/R^2\right)} = \text{const.}$$

(C)

thus $x(t) = \frac{1}{2}ct^2$

4.



only one coordinate is needed per pulley system because the rope lengths are fixed

$x_1 \rightarrow l - x$

$x_2 \rightarrow x + y$

$x_3 \rightarrow l - y + x$

$$T = \frac{1}{2}(4m)v_1^2 + \frac{1}{2}(3m)v_2^2 + \frac{1}{2}(m)v_3^2$$

$$= 2m\dot{x}^2 + \frac{3}{2}m(\dot{x}+\dot{y})^2 + \frac{1}{2}m(-\dot{y}+\dot{x})^2$$

$$= \frac{1}{2}m\left[4\dot{x}^2 + 3\dot{x}^2 + 3\dot{y}^2 + 6\dot{x}\dot{y} + \dot{y}^2 + \dot{x}^2 - 2\dot{x}\dot{y}\right]$$

$$T = \frac{1}{2}m\left[8\dot{x}^2 + 4\dot{y}^2 + 4\dot{x}\dot{y}\right]$$

$$U = -mgx \rightarrow U = U_1 + U_2 + U_3$$

$$= -4mg(l-x) - 3mg(x+y) - mg(l-y+x)$$

$$U = -5mgl + 4mgx - 3mgx - 3mgy + mgy - mgx$$

$$U = -5mgl - 2mgy$$

$$\mathcal{L} = T - U = \frac{1}{2}m\left[8\dot{x}^2 + 4\dot{y}^2 + 4\dot{x}\dot{y}\right] + 2mgy + 5mgl$$

→

$$\mathcal{L} = \tfrac{1}{2} m \left[8\dot{x}^2 + 4\dot{y}^2 + 4\dot{x}\dot{y} \right] + 2mgy + 5mgl$$

↑ constant

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\left[m8\dot{x} + 2\dot{y}m \right] = 0$$

$$4\ddot{x} + \ddot{y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2mg \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{d}{dt}\left[4\dot{y} + 2\dot{x} \right] m$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 2mg - 4m\ddot{y} - 2m\ddot{x} = 0$$

$$g = \ddot{x} + 2\ddot{y}$$

combining the two, $g = \ddot{x} + 2(-4\ddot{x}) = -7\ddot{x}$

$$\boxed{\ddot{x} = -g/7}$$

The $4m$ mass accelerates down at $g/7$.

The lower pulley is unbalanced and must accelerate, which means it's CM will accelerate. The top pulley's CM must also accelerate to get the correct force of tension.