Homework 1: Calculus of Variations, Taylor Ch. 6

I have tried to order these problems in a way that matches the order of presentation in lecture, and some reasonable increase in difficulty. I would recommend trying them in the order presented, but don't feel like you need to stay on one problem if you are very stuck!

1. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A.

2. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2. Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

3. Consider a right circular cylinder of radius R centered on the z axis. Find the equation giving ϕ as a function of z for the geodesic (shortest path) on the cylinder between two points with cylindrical polar coordinates $(R,\ \phi_1,\ z_1)$ and $(R,\ \phi_2,\ z_2)$. Describe the geodesic. Is it unique? By imagining the surface of the cylinder unwrapped and laid out flat, explain why the geodesic has the form it does.

4. A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere. Two hints to get you started: (1) the length element on the surface of a sphere is $ds = \rho(d\theta^2 + \sin^2\theta d\phi^2)^{1/2}$, and (2) consider how your function to optimize $f(y, y', x)$ depends on x, whatever x may turn out to be in this problem.

5. **Challenge problem!** Do your best on this one, but prioritize the other problems. I will grade for effort.

   You are given a string of fixed length ℓ with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x-axis in such a way as to enclose the maximum area between the string and the x-axis. Show that the required shape is a semicircle. The area enclosed is of course $\displaystyle\int y dx$, but show that you can rewrite this in the form $\displaystyle\int_0^\ell f ds$, where s denotes the distance measured along the string from 0, where $f = \sqrt{1 - y'^2}$, and y' denotes dy/ds. Since f does not involve the independent variable s explicitly, you can exploit Beltrami's identity.