Homework 1: Calculus of Variations, Taylor Ch. 6

I have tried to order these problems in a way that matches the order of presentation in lecture, and some reasonable increase in difficulty. I would recommend trying them in the order presented, but don't feel like you need to stay on one problem if you are very stuck!

1. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A.

   Surface area $A = 2\pi R^2 + 2\pi Rh$ is a function of two variables, R and h. We will use the constraint of constant volume $V = \pi R^2 h$ to constrain this problem and write A as a function of one variable. Then we can optimize on that single variable.

$$
\begin{aligned}
h = \frac{V}{\pi R^2} \longrightarrow A &= 2\pi R\left(\frac{V}{\pi R^2}\right) + 2\pi R^2 \\
&= \frac{2V}{R} + 2\pi R^2 \\
\frac{\partial A}{\partial R} &= \frac{-2V}{R^2} + 4\pi R = 0 \text{ to minimize} \\
\frac{2V}{R^2} &= 4\pi R \\
2V &= r\pi R^3 \\
R = \left(\frac{V}{2\pi}\right)^{-1/3} &= \left(\frac{\pi R^2 h}{2\pi}\right)^{-1/3} \\
R^3 &= R^2 h/2 \\
R &= h/2
\end{aligned}
$$

2. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2. Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

3. Consider a right circular cylinder of radius R centered on the z axis. Find the equation giving ϕ as a function of z for the geodesic (shortest path) on the cylinder between two points with cylindrical polar coordinates (R, ϕ_1, z_1) and (R, ϕ_2, z_2). Describe the geodesic. Is it unique? By imagining the surface of the cylinder unwrapped and laid out flat, explain why the geodesic has the form it does.

   To minimize the distance, we must first write out the distance element on the surface of a cylinder:

$$ds = \sqrt{dr^2 + r^2 d\phi^2 + dz^2}$$

   For a fixed cylinder, $r = R$ is constant so $dr = 0$. If we define $\phi(z)$ then $d\phi = \phi' dz$.

$$ds = \sqrt{R^2 d\phi^2 + dz^2} = \sqrt{R^2 \phi'^2 dz^2 + dz^2} = \sqrt{1 + R^2\phi'^2}\, dz$$

This leaves us to minimize the total path length $S = \int_{z_1}^{z_2} \sqrt{1 + R^2\phi'^2}dz$. We can write our general function $f(\phi, \phi', z) = \sqrt{1 + R^2\phi'^2}$ and use the Euler-Lagrange equation.

$$\frac{df}{d\phi} - \frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

We can see that $\frac{df}{d\phi} = 0$, so that leaves $\frac{df}{d\phi'} = c$ where c is some constant.

$$\begin{aligned}
\frac{df}{d\phi'} = c &= \frac{1}{2}\left(1 + R^2\phi'^2\right)^{-1/2} 2\phi' \\
&= \frac{\phi'}{1 + R^2\phi'^2} \\
\phi'^2 &= (1 + R^2\phi'^2)c \\
(1 - R^2c)\phi'^2 &= 1 \\
\phi'^2 &= \frac{1}{1 - R^2c} \longleftarrow \text{ that's a constant!}
\end{aligned}$$

We have shown that $\phi' = m$ where m is the constant left over at the end of our work above. Then we must have $\phi = mz + b$, a line! ϕ is linear in z. That's a helix. This solution is not unique because the helix could spiral either way. The minimum is the most direct helix, but we could get multiple rotations in and that would also be a solution. If we straighten out the cylinder, the helix looks like a straight line between the two points.

4. A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere. Two hints to get you started: (1) the length element on the surface of a sphere is $ds = \rho(d\theta^2 + \sin^2\theta d\phi^2)^{1/2}$, and (2) consider how your function to optimize $f(y, y', x)$ depends on x, whatever x may turn out to be in this problem.

Our goal is to find a path $\phi(\theta)$ that minimizes the distance between two points on a sphere. Our path length is $S = \int_1^2 \sqrt{\rho d\theta^2 + \rho\sin^2\theta d\phi^2}$. If we have some $\phi(\theta)$, then $d\phi = \frac{d\phi}{d\theta}d\theta = \phi' d\theta$. So, $S = \int_1^2 \sqrt{\rho}\sqrt{1 + \sin^2\theta\phi'^2}d\theta$. Then our function for the Euler-Lagrange equation is $f(\phi, \phi', \theta) = \sqrt{1 + \sin^2\theta\phi'^2}$.

The integral with respect to ϕ is zero, leaving

$$\begin{aligned}
\frac{\partial f}{d\phi'} &= C \\
\frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} &= C
\end{aligned}$$

At this point we can simplify our problem and choose our first point such that it is at the pole of the sphere. Then θ at this point is zero and the constant must be zero - since it's a constant it will always be zero.

The only way to make $\frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = 0$ as we change θ is to have $\phi' = 0$, which means that ϕ does not change with θ. This describes a longitudinal great circle on the surface of the sphere.

5. **Challenge problem!** Do your best on this one, but prioritize the other problems. I will grade for effort.

   You are given a string of fixed length ℓ with one end fastened at the origin O, and you are to place the string in the xy plane with its other end on the x-axis in such a way as to enclose the maximum area between the string and the x-axis. Show that the required shape is a semicircle. The area enclosed is of course $\int ydx$, but show that you can rewrite this in the form $\int_0^\ell fds$, where s denotes the distance measured along the string from 0, where $f = \sqrt{1-y'^2}$, and y' denotes dy/ds. Since f does not involve the independent variable s explicitly, you can exploit Beltrami's identity.

   The problem gives us quite a few hints, mainly that we need to recast our problem of finding a path $y(x)$ into finding a path $y(s)$, where s is the length of the path from the origin to any point x. We can start with the length element definition and go from there.

$$ds^2 = dx^2 + dy^2$$
$$dx = \sqrt{1-y'^2}ds \text{ where } y' = dy/ds$$

   If our orignal problem would have been to minimize $A = \int_0^x y(x)dx$, our new problem is to minimize $A = \int_0^\ell y(s)\sqrt{1-y'^2}ds$. Our general function is then $f(y,y',s) = y\sqrt{1-y'^2}$. This function does not depend explicitly on s, so we can use the Beltrami identity

$$\begin{aligned}
f - y'\frac{\partial f}{\partial y'} &= c \\
y\sqrt{1-y'^2} - y'\frac{\partial}{\partial y'}\left[y\sqrt{1-y'^2}\right] &= c \\
y\left[1-y'^2+y'^2\right] &= c\sqrt{1-y'^2} \\
y &= c\sqrt{1-y'^2} \\
y' = \frac{dy}{ds} &= \sqrt{1-y^2/c^2} \\
ds &= \frac{dy}{\sqrt{1-y^2/c^2}} \\
\int_0^\ell ds = \ell &= c\arcsin(y/c) \\
y &= c\sin(\ell/c)
\end{aligned}$$

   To integrate, I looked up a table of common integrals. You should make use of these.

   We now have a function $y(s)$, but we need a function $y(x)$. We will have to solve this as a

parametric equation and find a function $x(s)$. We can do that by going back to the definition of the length element

$$\begin{aligned} dx &= \sqrt{1 - y'^2}ds \\ x &= \int \sqrt{1 - y'^2}ds \\ &= \int \sqrt{1 - \cos^2(\ell/c)}ds \\ &= \int_0^\ell \sin(\ell/c)ds \\ &= -c\cos(\ell/c) + c \\ x &= c\left(1 - \cos(\ell/c)\right) \end{aligned}$$

If we combine our functions for x and y as $x^2 + y^2$ we get the equation of a circle. So, the path that minimizes the area under the path is a semicircle over the x-axis.