Homework 0: Review

1. Projectile Motion

   (a) Consider a projectile motion near the surface of the Earth, as shown in figure 1. Show that two times elapsed between points at the same elevation, t_A and t_B, as indicated on the figure are related by

$$t_A^2 - t_B^2 = \frac{8h}{g} \tag{1}$$

where h is the height between the two elevations.

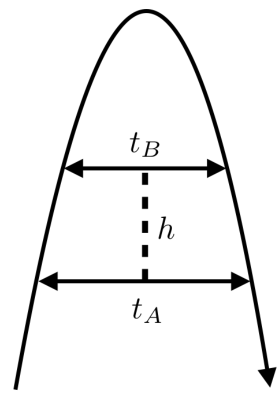

Figure 1: Projectile motion diagram for problem 2.

There are four times of interest in this problem: $t_{A,i}$ to $t_{A,f}$ and $t_{B,i}$ to $t_{B,f}$. We can use kinematic equations for each time point to relate the interval $t_A = t_{A,f} - t_{A,i}$ and $t_B = t_{B,f} - t_{B,i}$ to the height difference h.

$$\begin{aligned}
y_{A,i} &= 0 \\
y_{A,f} &= 0 = v_0 t_{A,f} - \frac{1}{2} g t_{A,i}^2 \\
y_{B,i} &= h = v_0 t_{B,i} - \frac{1}{2} g t_{B,i}^2 \\
y_{B,f} &= h = v_0 t_{B,f} - \frac{1}{2} g t_{B,f}^2
\end{aligned}$$

We can solve for $t_{A,f} = t_A = 2v_0/g$. The solutions for $t_{B,i}$ and $t_{B,f}$ should be the two solutions to the quadratic equation they share:

$$\begin{aligned}
t &= \frac{-v_0 \pm \sqrt{v_0^2 - 4(-g/2)(-h)}}{-g} \\
&= \frac{v_0 \pm \sqrt{v_0^2 - 2gh}}{g} \\
t_B &= \frac{v_0 + \sqrt{v_0^2 - 2gh}}{g} - \frac{v_0 \pm \sqrt{v_0^2 - 2gh}}{g} \\
&= \frac{2\sqrt{v_0^2 - 2gh}}{g} \\
t_A^2 - t_B^2 &= \frac{4v_0^2}{g^2} - \frac{4v_0^2 - 8gh}{g^2} \\
&= \frac{8h}{g}
\end{aligned}$$

(b) Show that this formula reduces to the standard time-of-flight result

$$t = \frac{2v_{yo}}{g} \tag{2}$$

where v_{yo} is the initial velocity in the y-direction, when $t_B = 0$.

This is just our original solution for t_A.

2. Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions :

(a) $F_x = F_0 + ct$

Integrating once from 0 to t gives $v(t) = F_0 t + \frac{1}{2}ct^2$. Integrating again from 0 to t gives $x(t) = \frac{1}{2}Ft^2 + \frac{1}{6}ct^3$.

(b) $F_x = F_0 \sin(ct)$

Integrating once from 0 to t gives $v(t) = \frac{F_0}{c}(1 - \cos(ct))$. Integrating again from 0 to t gives $x(t) = \frac{F_0}{c^2}(ct - \sin(ct))$.

(c) $F_x = F_0 \exp(ct)$, where F_0 and c are positive constants.

Integrating once from 0 to t gives $v(t) = \frac{F_0}{c}\left(e^{ct} - 1\right)$. Integrating again from 0 to t gives $x(t) = \frac{F_0}{c^2}\left(e^{ct} - ct - 1\right)$.

3. Find the potential energy function $V(x)$ for each of the following force functions : The potential is related to the force by $V(x) = -\int_0^x F_x dx$.

(a) $F_x = F_0 + cx$

$$V(x) = -\int_0^x F_x dx = -F_0 x - \frac{1}{2}cx^2$$

(b) $F_x = F_0 \exp(-cx)$

$$V(x) = -\int_0^x F_x dx = \frac{F_0}{c}\left(e^{cx} - 1\right)$$

(c) $F_x = F_0 \cos(cx)$, where F_0 and c are positive constants.

$$V(x) = -\int_0^x F_x dx = -\frac{F_0}{c}\sin(cx).$$

4. Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the force functions given in the previous problem.

(a) Assuming that F_x is the only force acting on the system and using the work-kinetic energy theorem

$$\begin{aligned} W = \int_0^x F_x dx &= \frac{1}{2}mv^2 \\ F_0 x + \frac{1}{2}cx^2 &= \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2}{m}F_0 x + \frac{1}{2}cx^2} \end{aligned}$$

(b)

$$\begin{aligned} W = \int_0^x F_x dx &= \frac{1}{2}mv^2 \\ \frac{F_0}{c}\left(1 - e^{cx}\right) &= \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2F_0}{mc}\left(1 - e^{cx}\right)} \end{aligned}$$

(c)

$$\begin{aligned} W = \int_0^x F_x dx &= \frac{1}{2}mv^2 \\ \frac{F_0}{c}\sin(cx) &= \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2F_0}{mc}\sin(cx)} \end{aligned}$$

5. (a) Show that the kinetic energy K of a particle with constant mass m moving at velocity v in 1D under the influence of a force F satisfies the equation

$$\frac{dK}{dt} = Fv \tag{3}$$

$$\begin{aligned}\frac{d}{dt}K &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\\ &= \frac{1}{2}m\frac{d}{dt}v^2\\ &= \frac{1}{2}m\left(2v\dot{v}\right)\\ &= \left(m\dot{v}\right)v = Fv\end{aligned}$$

(b) Now, consider the same situation as part a, except for a particle whose mass m is not constant. Show that the kinetic energy K satisfies

$$\frac{d(mK)}{dt} = Fp \tag{4}$$

Note that in the special case when m is constant, the result of part b reduces to that of part a.

$$\begin{aligned}\frac{d}{dt}(mK) &= \frac{d}{dt}\left(\frac{1}{2}m^2v^2\right)\\ &= \frac{1}{2}\frac{d}{dt}p^2\\ &= p\frac{dp}{dt} = Fp\end{aligned}$$