Phys 3210, Fall 2015 Practice Midterm #1 Sept. 18, 2015

Name: Prof. Neil

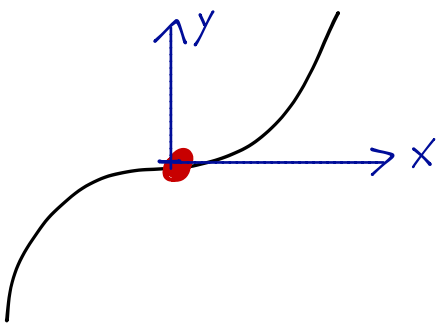
Read this first:

- Read through all the questions before you start! Starting with the questions you feel most confident in answering is always a good idea.
- Partial credit will be given whenever possible; show your work!
- If you don't know how to solve a problem in full, show whatever you do know; use dimensional analysis, write down relevant formulas, describe how you think the solution might be found.

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1. (16 points) Below are a number of mechanical systems. For each system, indicate the number of *degrees of freedom* that the underlined object(s) have. If there are any constraints on the motion of the red blocks, write down the appropriate *equations of constraint* using the provided coordinate system.

   (a) (4 points) A <u>mass</u> on a wire bent into the shape of a cubic polynomial, x^3.



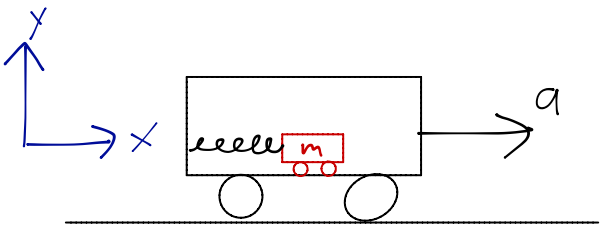
$d=2$, one constraint

$\Rightarrow n_{dof} = dN - m$

$= 2(1) - 1 = \boxed{1}$

$y = cx^3$

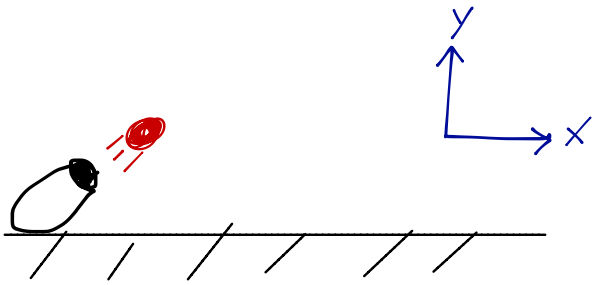
(b) (4 points) A <u>frictionless cart</u> attached to a spring, within a larger cart accelerating to the right with constant acceleration a.



Constraints: $y = 0$

$n_{dof} = 1$

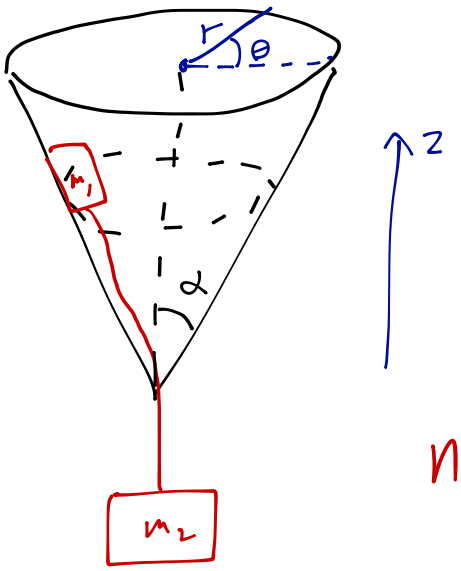
(c) (4 points) The motion of a <u>cannonball</u> launched with initial speed v_0, including gravity but not air resistance.



No constraints!

$n_{dof} = 2$

(d) (4 points) <u>Two masses</u> connected by a string of length ℓ, with one hanging vertically and the other constrained to move inside the surface of a cone with opening angle α.



Constraints:

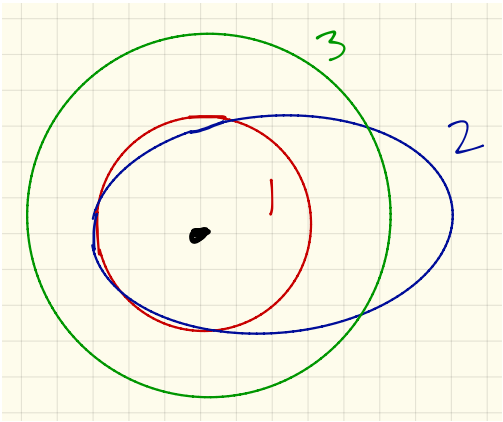
$r_2 = 0, \theta_2 = 0$ (hanging)

$z_1 = r_1 \cot\alpha$ (cone)

$\sqrt{r_1^2 + z_1^2} - z_2 = \ell$ (string)

$n_{dof} = 3(2) - 4 = \boxed{2}$

2. (16 points) A spacecraft with negligible mass orbits around the planet Mars. Consider the following set of three possible orbits:



1. A circular orbit, with radius 3 units.
2. An elliptic orbit, with periapsis at 3 units and apoapsis at 7 units.
3. A circular orbit, with radius 5 units.

(a) (6 points) Rank the orbits according to the duration of their *orbital period* τ, from shortest to longest.

$\tau \propto a^{3/2}$ $a_1 = 3$, $a_3 = 5$

$2a_2 = 3 + 7 = 10 \Rightarrow a_2 = 5$.

So, $\boxed{\tau_1 < \tau_2 = \tau_3}$.

(b) (10 points) All of these orbits are described by the equation of orbit

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}. \tag{1}$$

Determine the parameters c and ϵ for orbit 2.

$r_{min} = \frac{c}{1+\epsilon} = 3$, $r_{max} = \frac{c}{1-\epsilon} = 7$

$3(1+\epsilon) = 7(1-\epsilon)$

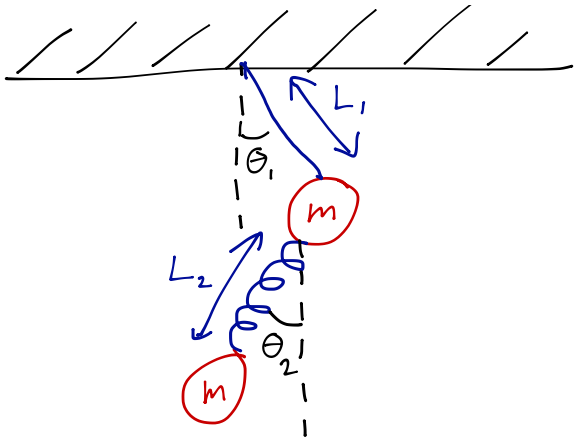
$10\epsilon = 4 \Rightarrow \boxed{\epsilon = 0.4}$

$c = 3(1+\epsilon) = 3(1.4)$

$\boxed{c = 4.2}$

(check: $c = 7(0.6) = 4.2$ ✓)

3. (20 points) Two equal masses m are connected to form a double pendulum, but the lower mass is connected with a spring of force constant k rather than a string. The unstretched length of the spring is ℓ.



(a) (5 points) Is this a **holonomic** system, that is, can we use the Lagrangian to derive its equations of motion? Explain.

Yes! One constraint, from string at top - holonomic (can be written as equation, no speed dependence.)

(b) (15 points) Write the **potential energy** U as a function of the coordinates θ_1, θ_2 and L_2 shown on the diagram.



$$y_1 = -L_1 \cos\theta_1$$

$$y_2 = y_1 - L_2\cos\theta_2$$

$$= -L_1\cos\theta_1 - L_2\cos\theta_2$$

$$U = mgy_1 + mgy_2 + \frac{1}{2}k(L_2-\ell)^2$$

$$U = -2mgL_1\cos\theta_1 - mgL_2\cos\theta_2 + \frac{1}{2}k(L_2-\ell)^2 + (\text{const.})$$

4. (20 points) Find the path $y(x)$ which extremizes the functional

$$J[y] = \int_0^1 dx\ \sqrt{x}\sqrt{1+y'}, \tag{2}$$

subject to the boundary conditions $y(0) = 0$ and $y(1) = 0$.

$F = \sqrt{x}\sqrt{1+y'}$ $\quad \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$

$0 = \frac{d}{dx}\left(\frac{\sqrt{x}}{2\sqrt{1+y'}}\right) \Rightarrow 2C\sqrt{1+y'} = \sqrt{x}$

$4C^2(1+y') = x$

$y' = \frac{x-4C^2}{4C^2} = \frac{x-d}{d}$

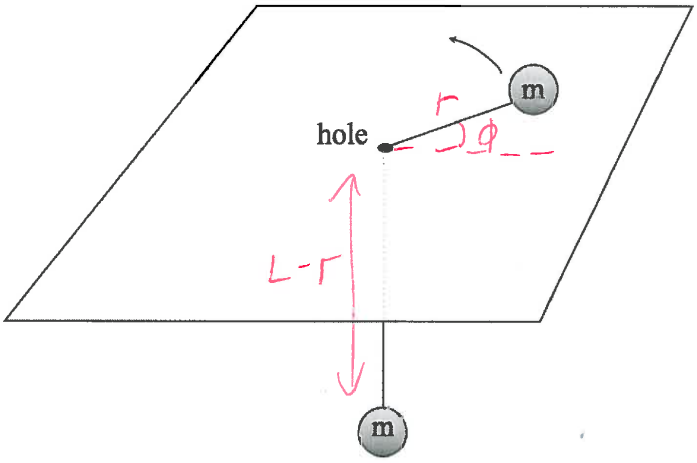
$y(x) = \int dx\ y' = \int dx\left(\frac{x}{d} - 1\right) = \frac{x^2}{2d} - x + y_0$

$y(0) = 0: \ y_0 = 0.$

$y(1) = 0: \ \frac{1}{2d} - 1 = 0 \Rightarrow d = \frac{1}{2}.$

$\Rightarrow y(x) = x^2 - x.$

5. (28 points) A ball of mass m slides without friction across a smooth, horizontal table. A second ball, also of mass m, is connected to the first by a massless, extensionless rope of length L. The second ball is suspended vertically below the table, and does not move parallel to the table, only up and down. However, the *first* ball is allowed to move freely in two dimensions on the surface of the table.



For this problem, we'll use the polar coordinates (r, ϕ) of the first ball as our generalized coordinates, as shown.

(a) (6 points) Write the Lagrangian for this system, in terms of the GCs r and ϕ. (You may omit any constant terms.)

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\dot{r}^2$$

$$U = +mgh = -mg(L - r)$$

$$\mathcal{L} = T - U$$

$$= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr + \cancel{mgL} \quad \text{const.}$$

(b) (12 points) Set up the Euler-Lagrange equations, and solve for the accelerations $\ddot{r}$ and $\ddot{\phi}$. (You do not need to solve these differential equations.)

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$-mg + mr\dot{\phi}^2 = 2m\ddot{r} \qquad 0 = \frac{d}{dt}(mr^2\dot{\phi})$$

$$\Rightarrow \ddot{r} = \frac{1}{2}r\dot{\phi}^2 - \frac{1}{2}g \qquad 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = 0$$

$$\Rightarrow \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}$$

(c) (10 points) Based on your result for $\ddot{r}$, are there any conditions under which the position of the hanging ball will stay fixed? If so, describe the corresponding motion of the ball on the table.

$$\ddot{r} = \frac{1}{2}r\dot{\phi}^2 - \frac{1}{2}g$$

$= 0$ if $\dot{\phi}^2 = g/r$

circular motion w/ const. radius $R = g/\dot{\phi}^2$ (and $\dot{\phi} = \text{const.}$)

END OF EXAM. Make sure your name is written on the front!