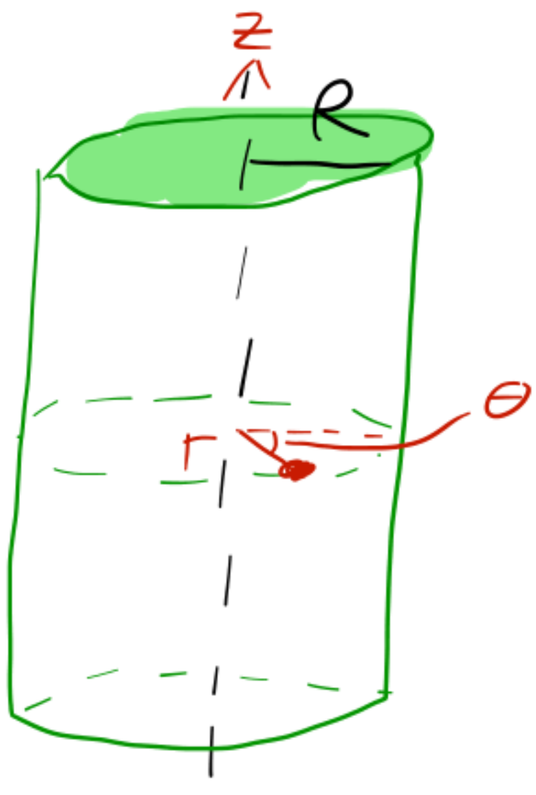
1. Below are a number of mechanical systems. For each system, indicate the number of *degrees of freedom* that the underlined dynamical object(s) have. If there are any constraints on the motion of the dynamical objects, write down the appropriate *equations of constraint* using the provided coordinate system.

   (a) **(5 pts)** A particle constrained to move on the surface of a cylinder of radius R. (Use cylindrical coordinates (r, θ, z).)



z not constrained

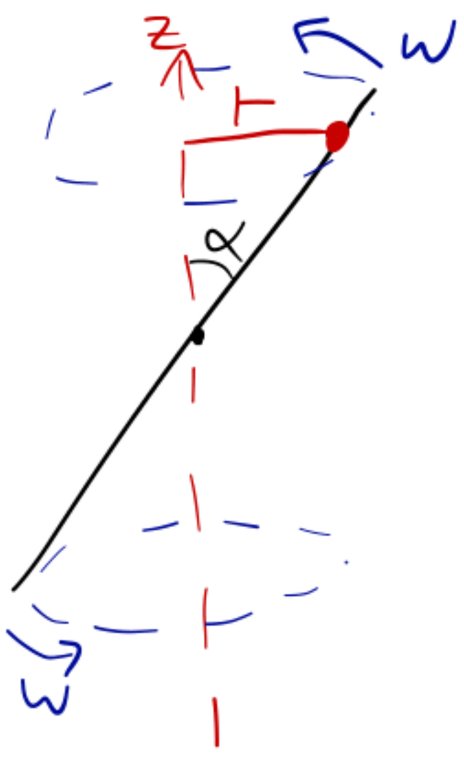
$x^2 + y^2 = R^2$ or $r = R$

θ not constrained

3 dim − 1 constraint = 2 d.o.f.

   (b) **(5 pts)** A bead is threaded on a straight wire, which is tilted at angle α to the vertical and spinning around the vertical axis with constant angular speed ω.

   (Use cylindrical coordinates (r, θ, z).)



1. $z = r \tan \alpha$

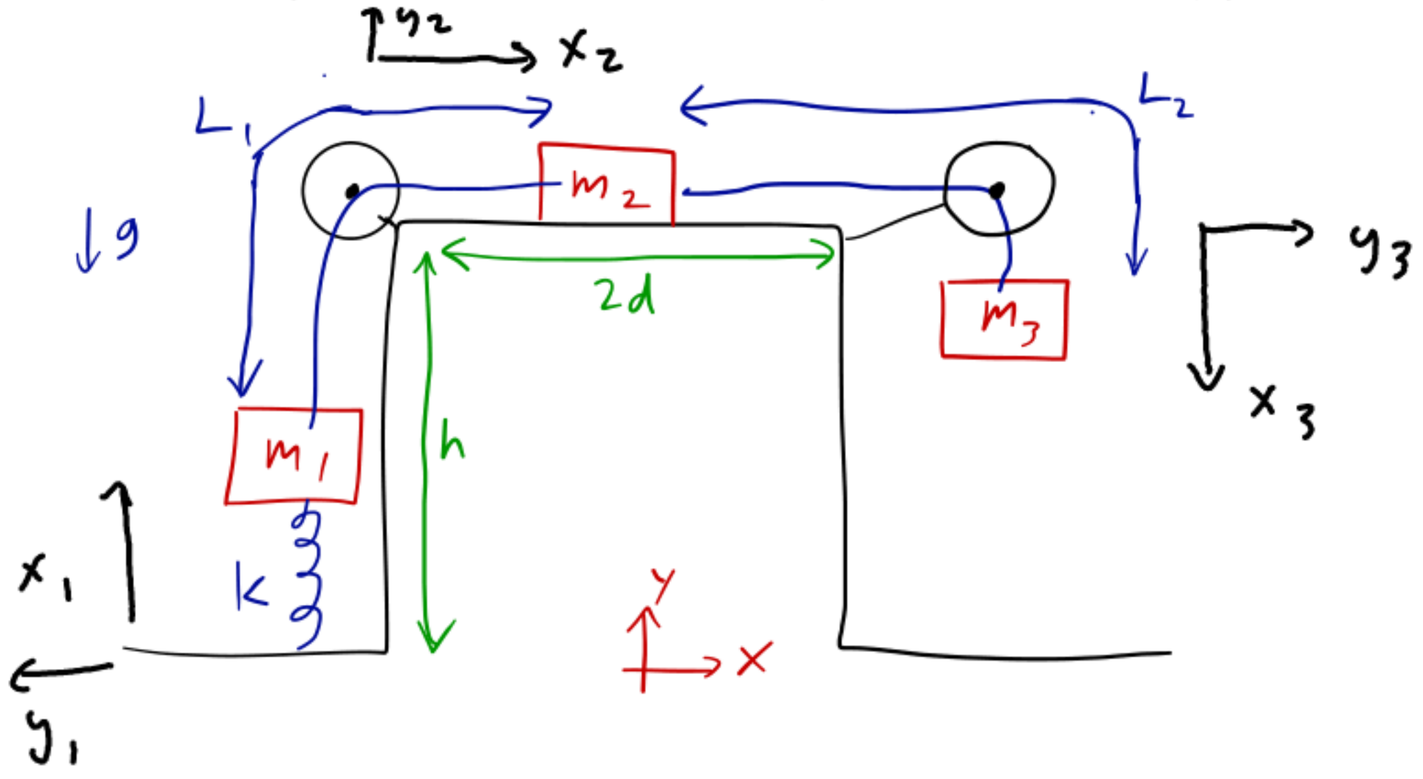
r not constrained

2. $\dot{\theta} = \omega$

3 dim − 2 constraint = 1 d.o.f.

(c) **(Extra Credit! 5pts)** Three blocks are arranged as pictured: block 2 sits exactly in the center of a frictionless table (height h and width $2d$), with blocks 1 and 3 hanging *vertically* over the edge supported by ropes of lengths L_1 and L_2. A spring connects block 1 to the floor; the unstretched length of the spring is ℓ. *The system begins at rest.*

(Use Cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$.)



1. $x_2 + h - x_1 = L_1$

2. $x_2 + L_2 - x_3 = 2d$

3. x_1 vertical $(y_1 = 0)$

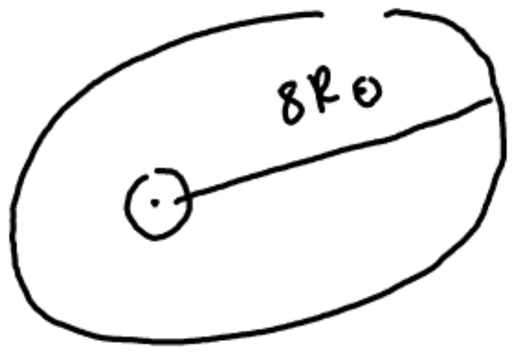
4. x_2 on table $(y_2 = 0)$

5. x_3 vertical $(y_3 = 0)$

3×2 dim $-$ 5 constraints $=$ 1 d.o.f.

2. An asteroid of mass m is traveling in an elliptical orbit around the Sun (mass $M_\odot \gg m$), with eccentricity $\epsilon = 0.8$. The asteroid begins at aphelion (maximum distance from the Sun), which occurs at a distance of $r_{\max} = 8R_\odot$ from the center of mass, where $R_\odot$ is the radius of the Sun.

   (a) **(10pts)** Will the asteroid fall into the Sun, that is, does its orbit cross $r = R_\odot$ anywhere?



$\epsilon = 0.8$

Is $r_{min} < R_\odot$?

$$r_{max} = \frac{c}{1-\epsilon} \qquad r_{min} = \frac{c}{1+\epsilon}$$

$$r_{max} = r_{min} = \frac{1+\epsilon}{1-\epsilon} \longrightarrow r_{min} = r_{max}\,\frac{1-\epsilon}{1+\epsilon}$$

$$= 8R_\odot \cdot \frac{0.2}{1.8} = \frac{8}{9} R_\odot$$

that is smaller than $R_\odot$, so it will fall into the sun.

   (b) **(10pts)** If the asteroid <u>does not</u> fall into the Sun, what is its speed at perihelion (closest approach)? If it <u>does</u> fall into the Sun, by how much does the total orbital energy E have to increase for the asteroid to instead escape the solar system completely?

If not: $v = \sqrt{\frac{\gamma}{\mu c}}\sqrt{\varepsilon^2 \sin^2\varphi + c^2/r^2}$ ($\varphi = 0$)

$c = r_{max}\cdot(1-\epsilon)$

$= 8R_\odot \cdot 0.2$

$= 1.6\,R_\odot$

$r_{min} = \frac{c}{1+\epsilon} = 1.6\,R_\odot / 1.8 =$

$v = \sqrt{\frac{\gamma}{\mu \cdot 1.6 R_\odot}}\sqrt{1.6^2/1.92^2}$

If fall: $E = -GMm/2a$

$a = \frac{1}{2}(r_{min} + r_{max})$

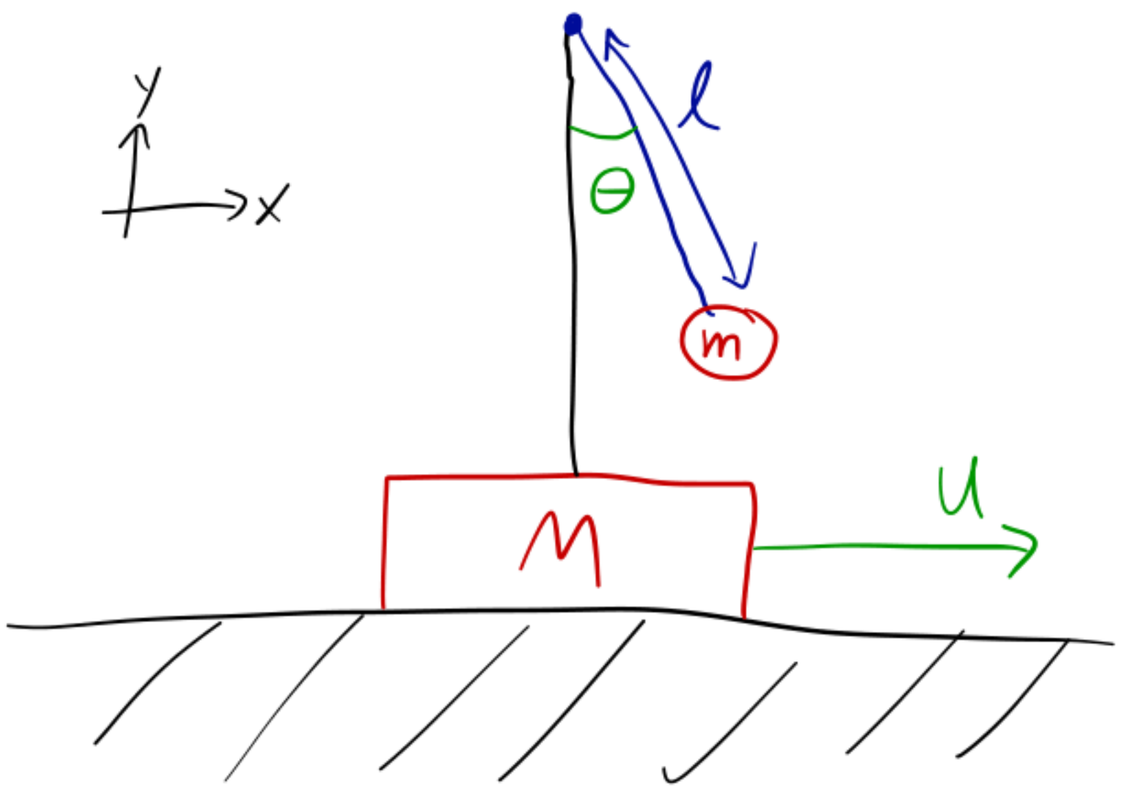
$= \frac{40}{9} R_\odot$

to escape, need $E_{tot} \geq 0$

add at least

$E = \frac{GMm}{\frac{80}{9} R_\odot}$

3. A pendulum of mass m and length ℓ is mounted on top of a block of mass M, which in turn is sitting on top of a frictionless surface.



As always, it's best to start in a Cartesian coordinate system (the xy system as shown), and then convert to generalized coordinates. We take the origin of the xy coordinate system to be at the initial position of the block. Thus, if (x_b, y_b) are the coordinates of the block and (x_p, y_p) the coordinates of the pendulum bob, then I can write some of the coordinates as

$$x_b = u,$$
$$y_b = 0,$$
$$y_p = h - \ell\cos\theta,$$

where h is the height of the top of the support, where the pendulum is hanging from.

(a) **(5pts)** Finish the coordinate change: write x_p in terms of θ and u.

$$x_p = x_b + \ell\sin\theta = u + \ell\sin\theta$$

(b) **(5pts)** The kinetic energy for this system in the original Cartesian coordinates is

$$T = \frac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2) + \frac{1}{2}M(\dot{x}_b^2 + \dot{y}_b^2).$$

Re-express T in terms of the generalized coordinates θ and u.

$x_p = u + l\sin\theta$

$\dot{x}_p = \dot{u} + l\cos\theta\,\dot{\theta}$

$y_p = h - l\cos\theta$

$\dot{y}_p = l\sin\theta\,\dot{\theta}$

$x_b = u \rightarrow \dot{x}_b = \dot{u}$

$y_b = 0 \rightarrow \dot{y}_b = 0$

$$T = \tfrac{1}{2}m\left[(\dot{u} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right] + \tfrac{1}{2}M\dot{u}^2$$

$$T = \tfrac{1}{2}m\left[\dot{u}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{u}l\cos\theta\,\dot{\theta} + l^2\sin^2\theta\,\dot{\theta}^2\right] + \tfrac{1}{2}M\dot{u}^2$$

$$T = \tfrac{1}{2}(m+M)\dot{u}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 + ml\dot{u}\cos\theta\,\dot{\theta}$$

(c) **(5pts)** Now find the potential energy U in terms of u and θ, and use it to write the Lagrangian $\mathcal{L}(u, \dot{u}, \theta, \dot{\theta})$ for this system.

$$U = mgy = mg(h - l\cos\theta)$$

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \tfrac{1}{2}(m+M)\dot{u}^2 + \tfrac{1}{2}ml^2\dot{\theta}^2 + ml\dot{u}\cos\theta\,\dot{\theta} - mgh + mgl\cos\theta$$

(d) **(10pts)** Aside from total energy, this system has another *constant of the motion*, that is to say, from the Euler-Lagrange equation for one of the two coordinates you should be able to identify some quantity P such that $dP/dt = 0$. What is P, in terms of θ and u? Explain what this quantity is physically.

E-L in u:

$$\frac{\partial \mathcal{L}}{\partial u} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) = 0$$

$\frac{\partial \mathcal{L}}{\partial \dot{u}}$ represents a conserved quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{u}} = (m+M)\dot{u} + ml\cos\theta\,\dot{\theta} = P$$

this is the total linear momentum of the system

END OF EXAM. Make sure your name is written on the front!