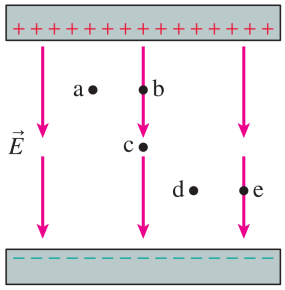
Practice Problems: Electric Potential

Subskills

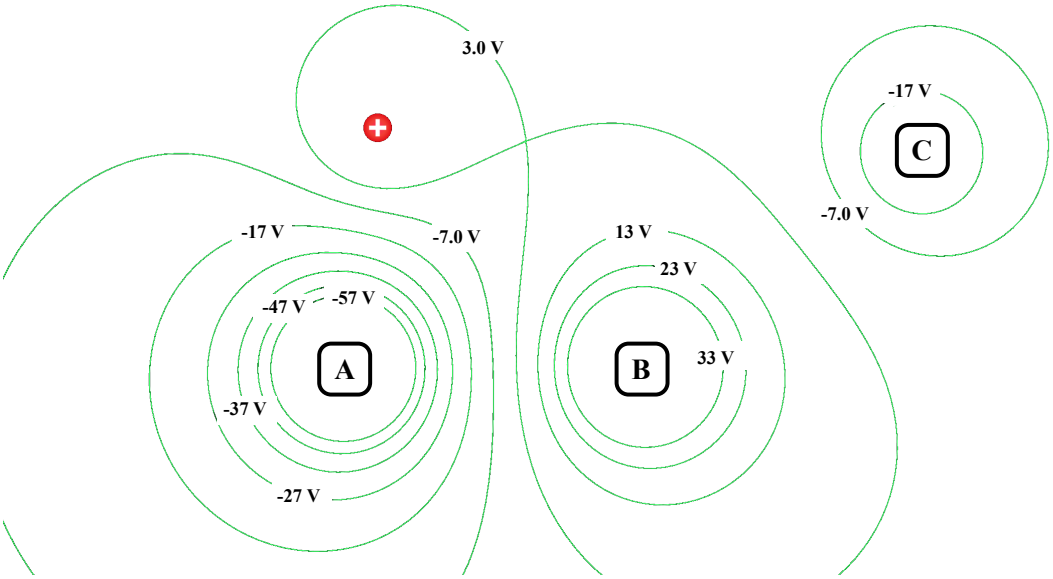
1. Two identical positive charges $+q$ are separated by distance r. The potential energy of this two-charge system is
   (a) $U = kq^2r^2$
   (b) $U = k\dfrac{q}{r}$
   (c) $U = k\dfrac{q^2}{r}$ ★
   (d) $U = k\dfrac{q}{r^2}$
   (e) $U = k\dfrac{q^2}{r^2}$
2. The potential at distance r from a point charge Q is
   (a) $V = k\dfrac{Q}{r}$ ★
   (b) $V = k\dfrac{Q}{r^2}$
   (c) $V = kQr$
   (d) $V = \dfrac{Qr}{k}$
3. Which of the following is true about electric potential (V) and electric potential energy (U)?
   (a) Both depend on the test charge.
   (b) Potential (V) depends only on the source charges; potential energy (U) depends on both source and test charges. ★
   (c) Potential energy (U) depends only on the source charges; potential (V) depends on both source and test charges.
   (d) Neither depends on the test charge.
4. If the distance between two like charges is doubled, the potential energy of the system
   (a) Doubles.
   (b) Halves. ★
   (c) Becomes one-quarter of the original.
   (d) Remains the same.

5. Rank in order, from largest to smallest, the potentials V_a to V_e at the points a to e. Note: the $\vec{E}-$field between 2 large parallel plates with equal and opposite surface charge Q is constant.

(a) $V_a = V_b < V_c < V_d = V_e$
(b) $V_a = V_b > V_c > V_d = V_e$★
(c) $V_a < V_b = V_c < V_d < V_e$
(d) $V_a > V_b = V_c > V_d > V_e$



6. The figure below shows equipotential lines for an arrangement of four charges. The red charge shown has charge $+q$. The three other charges have charge $-4q$,$+3q$, and $-q$. Match the locations A, B and C with the appropriate charge.



A: -4q

B: +3q

C: -q

In lecture, we learned that the $E-$field can be calculated from the potential V by taking the gradient: $\vec{E} = -\nabla V$ or $\vec{E}_s = -\frac{dV}{ds}\hat{s}$ in one dimension. Fill in the blanks for the following two questions with the given responses "high" or "low".

(a) high, high

(b) high, low

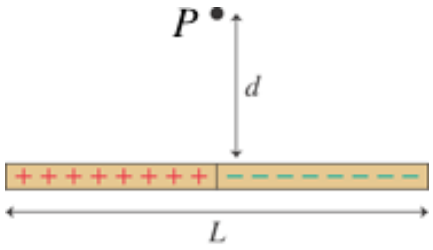
(c) low, high

(d) low, low

7. The gradient points from ___ potential to ___ potential. c
8. The electric field points from ___ potential to ___ potential.b

9. The left half of the uniformly charged rod in the figure below carries $+Q$, and the right half carries $-Q$. What is the electric potential at point P?

(a) $V < 0$
(b) $V = 0$ ★
(c) $V > 0$



10. Would it require substantially more calculation to calculate the electric field or the electric potential at point P in the previous problem?

(a) Electric field

(b) Electric potential★

11. If we brought a point particle with charge $+e$ to point P from infinity, which of the following is true about the change in the particle's electric potential energy ΔU?

(a) $\Delta U < 0$

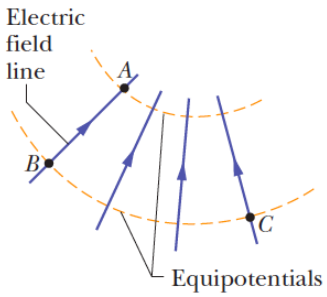
(b) $\Delta U = 0$★

(c) $\Delta U > 0$

(d) ΔU depends on the path the particle takes

When an electron moves from A to B along an electric field line the electric field does $W = 1.67 \times 10^{-19}$J amount of work on it. The corresponding change in the electric potential from A to B is $V_B - V_A = 1.00$ V. Find the potential differences shown below; possible answers are 1 V, -1V and 0V.

(a) 1 V
(b) 0 V
(c) -1 V



12. $V_B - V_A$

    a

13. $V_C - V_A$

    a

14. $V_A - V_C$

    c

15. $V_C - V_B$

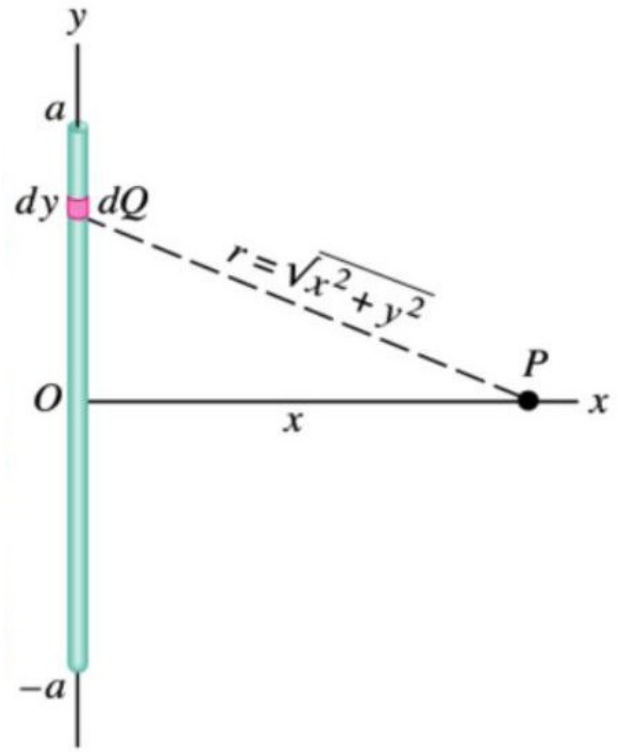
    b

16. $V_B - V_C$

    b

Problems

1. (10 Points) A thin non-conducting rod of length $2a$ lies along the y-axis and has a uniform linear charge density λ, as shown in the figure.



Calculate the electric potential V due to the rod at point P, which lies a perpendicular distance x from the center of the rod. Note that you will need the following integral to solve this problem:

$$\int \frac{dy}{\sqrt{c^2+y^2}} = \ln\left[y + \sqrt{c^2+y^2}\right],$$

where c is a constant.

For a continuous charge distribution, the electric potential at point $P = (x, y, z)$ in space is given by the general expression,

$$V_P = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r},$$

where $\varepsilon_0 = 8.854 \times 10^{-12}\dfrac{\mathrm{C}}{\mathrm{N}\cdot\mathrm{m}}$, and r is the radial distance between point P and the element of charge dq in the charge distribution. The integral must be taken over the entire charge distribution. Note that our expression for the potential assumes that the potential at $V(r = \infty) = 0$. In this problem, the charge distribution is a uniform rod, which has a charge per unit length λ and lies along the y-axis. The corresponding element of charge dq for this distribution is,

$$dq = \lambda dy,$$

where dy is the element of length along the rod. The limits of integration are from $y = -a$ to the end of the rod, $y = a$. The radial distance r between point $P = (x, 0, 0)$ and the element of

charge dq is,

$$r = \sqrt{x^2 + y^2},$$

where y is the distance from the origin to dy.

$$V_P = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int_{y=-a}^{y=a} \frac{\lambda dy}{\sqrt{x^2+y^2}} = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\int_{y=-a}^{y=a} \frac{dy}{\sqrt{x^2+y^2}}.$$

We can evaluate this integral using the indefinite integral that the problem gives us,

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\ln\left[y+\sqrt{x^2+y^2}\right]\Big|_{y=-a}^{y=a} = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\left(\ln\left[a+\sqrt{a^2+x^2}\right] - \ln\left[-a+\sqrt{a^2+x^2}\right]\right).$$

It is important to notice that the potential is positive, which we should expect for a positive charge distribution.
We can simplify this in the following way:

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\ln\left[\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\right]$$

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\ln\left[\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a}\cdot\frac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}+a}\right]$$

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)\ln\left[\frac{\left(\sqrt{a^2+x^2}+a\right)^2}{x^2}\right]$$

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)2\ln\left[\frac{\sqrt{a^2+x^2}+a}{x}\right]$$

Remark: As expected, our expression for the potential is more complicated than the potential for a point charge. However, we can understand V_P better by examining how it behaves in various limits. First, we can check that the units of our expression are volts (V):

$$[V_P] = \left[\left(\frac{\lambda}{4\pi\epsilon_0}\right)2\ln\left[\frac{a+\sqrt{a^2+x^2}}{x}\right]\right] = \frac{[\lambda]}{[\epsilon_0]} = \frac{\frac{\mathrm{C}}{\mathrm{m}}}{\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}} = \frac{\mathrm{N}\cdot\mathrm{m}}{\mathrm{C}} = 1\ \mathrm{V}.$$

Now let's consider what happens in the limit that point P is extremely far away from the line of charge, $x \gg a$. When we are extremely far about from the uniform rod, we should intuitively expect for the rod to become a small isolated point of charge $Q = 2\lambda a$. Dividing a by x and using the Taylor Series Expansion $\ln\left(z+\sqrt{z^2+1}\right) \approx z$ for small z, we have

$$V_P = \left(\frac{\lambda}{4\pi\epsilon_0}\right)2\ln\left[\frac{a}{x}+\sqrt{\left(\frac{a}{x}\right)^2+1}\right] \approx \left(\frac{\lambda}{4\pi\epsilon_0}\right)2\left(\frac{a}{x}\right) = \frac{Q}{4\pi\epsilon_0 x}.$$

This is precisely the electric potential for a point charge at a distance d around from the charge Q. Note that finding the potential in this limit is not required for this problem, but an extremely useful way to understand the final answer.

2. (15 points) When the electric potential depends on more than one variable, we need to use the gradient operator $\vec{\nabla}$ to find the gradient. The gradient operator turns a scalar function $V(x, y, z)$ into a vector:

$$\vec{\nabla}V(x, y, z) = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \tag{1}$$

For example, the gradient of $V(x, y, z) = x^2 + y^2 + z^2$ is $\vec{\nabla}V(x, y, z) = \Big(2x, 2y, 2z\Big)$. The gradient points perpendicular to the equipotentials and towards the point of steepest ascent. The electric field is most generally found from the potential through $\vec{E} = -\vec{\nabla}V(x, y, z)$. Find the electric field from the following electric potentials:

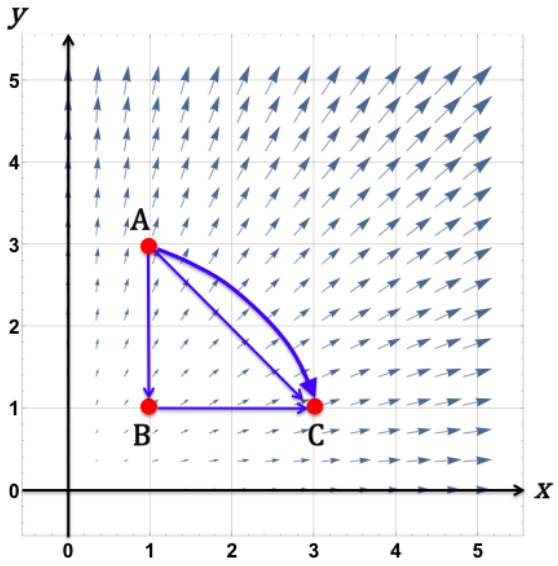
(a) $V(x, y, z) = x + y + z$

$-\hat{x} - \hat{y} - \hat{z}$

(b) $V(z) = \frac{\eta}{2\epsilon_0}(\sqrt{z^2 + R^2} - z)$, where η and ϵ_0 are constants.

$-\frac{\eta}{2\epsilon_0}\left(\frac{z}{\sqrt{z^2 + R^2}} - 1\right)\hat{z}$

3. (15 points) Consider the radial electric field $\vec{E} = \beta(x\hat{x} + y\hat{y})$ and the points $A = (1, 3, 0)$, $B = (1, 1, 0)$, and $C = (3, 1, 0)$, as shown in the figure below.



(a) Calculate the potential difference between points A and B.

$$\begin{aligned}\Delta V_{AB} &= V_B - V_A \\ &= -\int_A^B \vec{E} \cdot d\vec{s} \\ &= -\int_3^1 \beta(x\hat{x} + y\hat{y}) \cdot dy\hat{y} \\ &= -\beta \frac{y^2}{2}\Big|_3^1 \\ \Delta V_{AB} &= 4\beta\end{aligned}$$

OR

$$\begin{aligned}\Delta V_{AB} &= V_B - V_A \\ &= -\int_A^B \vec{E} \cdot d\vec{s} \\ &= -\int_1^3 \beta(x\hat{x} + y\hat{y}) \cdot -dy\hat{y} \\ &= \beta \frac{y^2}{2}\Big|_1^3 \\ \Delta V_{AB} &= 4\beta\end{aligned}$$

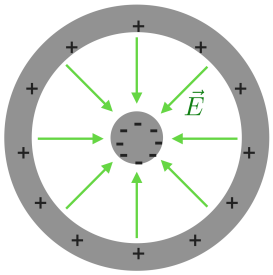
(b) Calculate the potential difference between points B and C.

-4β

(c) Calculate the potential difference between points A and C.

0β

4. The figure below shows a cross section of a device made from concentric **spherical** conductors. The inner sphere has charge $-Q$ and radius R_a. The outer shell has charge $+Q$ and inner radius R_b.



(a) Use Gauss's law to find the electric field in the region between the conductors, in the region $R_a < r < R_b$.

$$\oint \vec{E} \cdot \vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\implies E(4\pi r^2) = \frac{-Q}{\epsilon_0}$$

$$\implies E = -k\frac{Q}{r^2}.$$

(b) Use your result from part (a) to determine the electric potential difference between the conductors.

$$\Delta V = -\int \vec{E} \cdot d\vec{l} = kQ \int_{R_a}^{R_b} \frac{1}{r^2} dr = kQ \left(\frac{1}{R_a} - \frac{1}{R_b} \right).$$